DISS. ETH NO. 24788

Cohomology of Configuration Spaces

A thesis submitted to attain the degree of
Doctor of Sciences
(Dr. sc. ETH Zurich)

presented by

Christoph Schiessl
M. Sc. LMU Munich
born on 04.04.1990
citizen of Germany

accepted on the recommendation of

Prof. Dr. Rahul Pandharipande
Prof. Dr. Emanuele Delucchi

2017

Summary

For any topological space X, let

$$F_n(X) = \{(x_1, \dots, x_n) \in X^n | \; x_i \neq x_j\}$$

be the *ordered configuration space* of n distinct points in X. The symmetric group S_n acts on $F_n(X)$ by permuting the points and the quotient

$$C_n(X) = F_n(X)/S_n$$

is the *unordered configuration space*.

This thesis presents various explicit computations of cohomology groups of configuration spaces.

In chapter 1, we explain some background about configuration spaces and discuss the various methods that exist for computing their cohomology.

In chapter 2, we compute the rational cohomology groups of the unordered configuration spaces of the torus using a method of Félix and Thomas. These Betti numbers were previously unknown. This has been published as a preprint [Sch16].

In chapter 3, we describe classical calculations by Fuks and Vainshtein of the group $H^*(C_n(\mathbb{C}), \mathbb{Z})$ and show how they can be extended to $H^*(C_n(S^2), \mathbb{Z})$ using a cellular decomposition of Napolitano. For $\mathbb{Z}/p\mathbb{Z}$-coefficients, the cohomology of $C_n(S^2)$ has already been determined by Salvatore. However, our approach is more elementary and also works with integral coefficients.

In chapter 4, we compute the virtual Poincaré polynomials of the space of n distinct points on an elliptic curve with sum 0 by extending methods of Getzler. The result is new.

In chapter 5, we compare ordinary and virtual Poincaré polynomials for ordered and unordered configuration spaces of $\mathbb{C} \setminus k$ points. We apply different well-known approaches, however some of the explicit formulas seem not to be in the literature yet.

Zusammenfassung

Sei X ein topologischer Raum. Dann ist

$$F_n(X) = \{(x_1, \dots, x_n) \in X^n | \; x_i \neq x_j\}$$

der *geordnete Konfigurationsraum* von n verschiedenen Punkten auf X. Die symmetrische Gruppe S_n operiert auf $F_n(X)$ durch Permutation der Punkte und der Quotient

$$C_n(X) = F_n(X)/S_n$$

ist der *ungeordnete Konfigurationsraum.*

Diese Arbeit enthält verschiedene explizite Berechnungen von Kohomologiegruppen von Konfigurationsräumen.

In Kapitel 1 führen wir Konfigurationsräume ein und diskutieren die verschiedenen Methoden, um ihre Kohomologie zu berechnen.

In Kapitel 2 berechnen wir die rationalen Kohomologiegruppen des ungeordnenten Konfigurationsraumes eines Torus mit einer Methode von Félix und Thomas. Diese Bettizahlen waren vorher unbekannt. Das Kapitel wurde als Preprint veröffentlicht [Sch16].

In Kapitel 3 beschreiben wir klassische Berechnungen von $H^*(C_n(\mathbb{C}), \mathbb{Z})$ durch Fuks und Vainshtein und zeigen, wie diese mit einer zellulären Zerlegung von Napolitano auf $H^*(C_n(S^2), \mathbb{Z})$ erweitert werden können. Mit $\mathbb{Z}/p\mathbb{Z}$-Koeffizienten wurde die Kohomologie von $C_n(S^2)$ schon von Salvatore berechnet. Unser Ansatz ist jedoch elementarer und funktioniert auch mit ganzzahligen Koeffizienten.

In Kapitel 4 bestimmen wir die virtuellen Poincaré-Polynome des Raumes von n verschiedenen Punkten auf einer elliptischen Kurve mit Summe 0, indem wir Methoden von Getzler erweitern. Das Resultat ist neu.

In Kapitel 5 vergleichen wir gewöhnliche und virtuelle Poincaré-Polynome von geordneten und ungeordneten Konfigurationsräumen von $\mathbb{C} \setminus k$ Punkte. Wir wenden verschiedene bekannte Methoden an, einige der expliziten Formeln scheinen jedoch noch nicht in der Literatur vorhanden zu sein.

Acknowledgements

The author was supported by the grant ERC-2012-AdG-320368-MCSK.

I want to thank Karim Adiprasato, Filippo Callegaro, Gabriel Drummond-Cole, Emanuele Delucchi, Javier Fresán, Felix Janda, Giovanni Gaiffi, Emmanuel Kowalski, Dan Petersen, Paolo Salvatore, Johannes Schmitt, Junliang Shen, Qizheng Yin for very helpful discussions and especially Rahul Pandharipande for his very own way of invaluable support.

Contents

CHAPTER 1

Configuration Spaces and their Cohomology

1. Examples

In general, it is quite hard to describe the topology of configuration spaces, usually it is only possible if the number of points is small. Some examples:

(1) $F_1(X) = C_1(X) = X$
(2) After identification $\mathbb{R} \cong (0,1)$, any point in $C_n(\mathbb{R})$ is a set of n points on the unit interval, so $C_n(\mathbb{R})$ is a n-dimensional simplex and $F_n(\mathbb{R})$ is an union of $n!$ copies of $C_n(\mathbb{R})$.
(3) The fundamental group of $F_n(\mathbb{C})$ is Artin's braid group and all its higher homotopy groups vanish, so it is a classifying space.
(4) $F_2(\mathbb{R}^m) \cong \mathbb{R}^m \times \mathbb{R}^m \setminus 0$ via $(z_1, z_2) \mapsto (z_1, z_1 - z_2)$
(5) $C_2(\mathbb{C}) \cong \mathbb{R}^3 \times \mathbb{RP}^1$ via $(z_1, z_2) \mapsto (\frac{z_1 - z_2}{2}, |z_1 - z_2|, \mathbb{R}(z_1 - z_2))$.
(6) $F_2(S^m) \sim S^m$ via $(x_1, x_2) \mapsto \frac{x_1 - x_2}{|x_1 - x_2|}$, $x \mapsto (x, -x)$.
(7) $F_3(S^2) \cong F_3(\mathbb{CP}^1) \cong PGL(2, \mathbb{C}) \sim SO_3 \cong \mathbb{R}P^3$ via Moebius transformations

$$(z_1, z_2, z_3) \mapsto \frac{z_3 - z_2}{z_3 - z_1} \frac{z - z_1}{z - z_2}.$$

The group $PGL(2, \mathbb{C})$ retracts to $PSU(2, \mathbb{C}) \simeq SO_3$ via QR-decomposition.

2. Cohomology of ordered Configuration Spaces

Maybe the first computation of cohomology groups of configuration spaces was done by Arnold [Arn69], who determined $H^*(F_n(\mathbb{C}), \mathbb{Z})$. Forgetting the last point creates a map

$$F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C}),$$

which forms a fiber bundle with fiber $\mathbb{C} \setminus n-1$. The bundle has a section by adding a point "far away":

$$z_n = \frac{z_1 + \cdots + z_{n-1}}{n-1} + 2 \max_{1 \leq i,j \leq n-1} |z_i - z_j| + 1.$$

Looking at the spectral sequence of the fiber bundle $F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$, Arnold could show that the cohomology groups satisfy

$$H^*(F_n(\mathbb{C}), \mathbb{Z}) = H^*(F_{n-1}(\mathbb{C}), \mathbb{Z}) \otimes H^*(\mathbb{C} \setminus n-1, \mathbb{Z}).$$

Hence one can recursively conclude that the cohomology groups are torsion free and the Poincaré polynomials are

$$\sum \operatorname{rk} H^i(F_n(X), \mathbb{Z})\, t^i = (1+t)(1+2t)\cdots(1+(n-1)t).$$

Let $A(n)$ be the exterior algebra over $\mathbb{Z}$ with generators $\omega_{i,j}$ of degree 1 for $1 \le i \ne j \le n$ and relations

$$\omega_{i,j} = \omega_{j,i} \qquad\qquad \omega_{i,j}\omega_{j,k} + \omega_{j,k}\omega_{k,i} + \omega_{k,i}\omega_{i,j} = 0.$$

Theorem 2.1. *[Arn69] The identification*

$$w_{i,j} = \frac{1}{2\pi i}\frac{dz_i - dz_j}{z_i - z_j}$$

of generators defines an isomorphism

$$H^*(F_n(\mathbb{C}), \mathbb{Z}) \simeq A(n)$$

of algebras. An additive basis of $A(n)$ is given by all elements of the form

$$\omega_{i_1,j_1}\omega_{i_2,j_2}\cdots\omega_{i_p,j_p} \text{ where } 1 \le i_s < j_s \le n \text{ and } 1 \le j_1 < j_2 < \cdots < j_p \le n.$$

As suggested by Arnold, similar formulas describe the cohomology of the complement of hyperplanes in $\mathbb{C}^k$ in terms of the combinatorial structure of the hyperplanes [OS80].

Cohen and Taylor [CT93] extended Thm. 2.1 to $F_n(\mathbb{R}^m)$. Let $\omega \in H^{m-1}(F_2(\mathbb{R}^m), \mathbb{Z})$ be the image of the standard generator of $H^{m-1}(S^{m-1}, \mathbb{Z})$ under the homotopy equivalence

$$F_2(\mathbb{R}^m) \to S^{m-1},\ (z_1, z_2) \mapsto \frac{z_1 - z_2}{|z_1 - z_2|}.$$

Define

$$\pi_{i,j} \colon F_n(\mathbb{R}^m) \to F_2(\mathbb{R}^m),\ (x_1, \dots, x_n) \mapsto (x_i, x_j)$$

and

$$w_{i,j} = \pi_{i,j}^*(\omega) \in H^{m-1}(F_n(\mathbb{R}^m), \mathbb{Z}).$$

Theorem 2.2. *The cohomology ring $H^*(F_n(\mathbb{R}^m), \mathbb{Z})$ is generated by the elements $\omega_{i,j} = \omega_{j,i} \in H^{m-1}(F_n(\mathbb{R}^m), \mathbb{Z})$ for $1 \le i < j \le n$ with the only relations $\omega_{i,j}\omega_{j,k} + \omega_{j,k}\omega_{k,i} + \omega_{k,i}\omega_{i,j} = 0$.*

An extensive discussion of the homotopy and homology of $F_n(\mathbb{R}^k)$ and $F_n(S^k)$ can also be found in [FH01].

Let X be a smooth, projective variety over $\mathbb{C}$ of complex dimension l. Totaro [Tot96] was able to show that the cohomology ring $H^*(F_n(X), \mathbb{Q})$ is determined by the cohomology algebra $H^*(X, \mathbb{Q})$. He used the Leray spectral sequence associated to the compactification

$$F_n(X) \to X^n,$$

which degenerates after the first non-trivial differential in this case. Define

$$p_i \colon X^n \to X, (x_1, \dots, x_n) \mapsto x_i$$

and

$$p_{i,j} \colon X^n \to X^2, (x_1, \dots, x_n) \mapsto (x_i, x_j).$$

Let $\Delta \in H^{2l}(X^2)$ be the class of the diagonal.

Theorem 2.3. *Let $E(n)$ be the free graded $\mathbb{Q}$-algebra $H^*(X^n)[\omega_{i,j}]$ with generators $\omega_{i,j}$ of degree $2l-1$ for $1 \le i \ne j \le n$ and the relations*

- $\omega_{i,j} = \omega_{j,i}$
- $\omega_{i,j}^2 = 0$

- $\omega_{i,j}\omega_{j,k} + \omega_{j,k}\omega_{k,i} + \omega_{k,i}\omega_{i,j} = 0$ *for* i, j, k *distinct.*
- $p_i^*(\alpha)\omega_{i,j} = p_j^*(\alpha)\omega_{i,j}$ *for* $i \neq j$, $\alpha \in H^*(X)$

A differential d *can be defined by*

$$d_{|H^*(X^n)} = 0, \qquad d\omega_{i,j} = p_{i,j}^*\Delta.$$

Then $(E(n), d)$ *computes the rational cohomology ring of* $F_n(X)$*.*

$$H^*(F_n(X), \mathbb{Q}) \simeq H^*(E(n), d).$$

The action of $\sigma \in S_n$ *on* $H^*(F_n(X), \mathbb{Q})$ *is given by the obvious action on* $H^*(X^n)$ *and* $\sigma(\omega_{i,j}) = \omega_{\sigma(i),\sigma(j)}$*.*

Observe that $E(n)$ could be seen as a global version of the algebra $A(n)$ from theorem 2.1. A similar, but more complicated DGA model of $H^*(F_n(X), \mathbb{Q})$ was given by Fulton and MacPherson [FM94] using the Fulton-MacPherson compactification of $F_n(X)$. Kriz could algebraically simplify it to $E(n)$ [Kri94].

In practice the cohomology of the algebra $E(n)$ is rather tedious to compute – especially for bigger n. It can also be used for the unordered configuration space by using the transfer isomorphism

$$H^*(C_n(X), \mathbb{Q}) = H^*(F_n(X), \mathbb{Q})^{S_n}.$$

We will discuss this in more detail in the next chapter. Examples are:

Theorem 2.4. *[Aza15] The Poincaré polynomials of configuration spaces of two points on a Riemann surface* Σ_g *of genus* g *are*

$$P(C_2(\Sigma_g)) = 1 + 2gt + (2g^2 - g)t^2$$
$$P(F_2(\Sigma_g)) = 1 + 4gt + (4g^2 + 1)t^2 + 2gt^3.$$

Theorem 2.5. *[Aza15], [BMP05] The Poincaré polynomials of 3 points on a Riemann surface* Σ_g *of genus* $g \geq 2$ *are*

$$P(C_3(\Sigma_g)) = 1 + 2gt + (2g^2 - g)t^2 + \frac{1}{3}(4g^3 - g + 3)t^3 + 2gt^4$$
$$P(F_3(\Sigma_g)) = 1 + 6gt + 12g^2t^2 + (8g^3 + 2g^2 + g + 1)t^3 + (2g^2 + 3g)t^4.$$

Theorem 2.6. *[Aza15] For genus 1, we have*

$$P(C_3(\Sigma_1)) = 1 + 2t + 3t^2 + 4t^3 + 2t^4$$
$$P(F_3(\Sigma_1)) = (1 + t)^2(1 + 4t + 5t^2)$$
$$P(F_4(\Sigma_1)) = 1 + 2t + 3t^2 + 5t^3 + 4t^4 + t^5.$$

Another result of similar type is [AB14], where the cohomology groups of $F_3(\mathbb{CP}^m)$ and $C_3(\mathbb{CP}^m)$ are computed.

3. Unordered Configuration Spaces

Arnold interprets the points of $C_n(\mathbb{C})$ as monic degree n polynomials with complex coefficients without multiple roots via

$$(z_1, \ldots, z_n) \mapsto (z - z_1)\cdots(z - z_n) = z^n + \lambda_{n-1}z^{n-1} + \cdots + \lambda_1 z + \lambda_0.$$

So $C_n(\mathbb{C})$ can be identified with the complement $\mathbb{C}^n \setminus \Delta$ of the discriminant $\Delta = \prod_{i\neq j}(\lambda_i - \lambda_j)$. Arnold could compute the cohomology of $C_n(\mathbb{C})$ by applying Alexander duality to the compactification $C_n(\mathbb{C}) \simeq \mathbb{C}^n \setminus \Delta \subset \mathbb{C}^n \subset S^{2n}$ and using filtrations of the set of polynomials by the multiplicities of their roots.

Theorem 3.1. *[Arn70] The cohomology groups $H^*(C_n(\mathbb{C}), \mathbb{Z})$ satisfy the following properties:*

(1) *(Finiteness) All cohomology groups are finite except $H^0(C_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$ and $H^1(C_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$ for $n \geq 2$.*
(2) *(Vanishing) $H^i(C_n(\mathbb{C}), \mathbb{Z}) = 0$ for $i \geq n$.*
(3) *(Recurrence) $H^i(C_{2n+1}(\mathbb{C}), \mathbb{Z}) = H^i(C_{2n}(\mathbb{C}), \mathbb{Z})$*
(4) *(Stability) For increasing n, the cohomology groups stabilize:*

$$H^i(C_n(\mathbb{C}), \mathbb{Z}) = H^i(C_{n+1}(\mathbb{C}), \mathbb{Z}) \text{ if } n \geq 2i - 2.$$

The isomorphism is induced by pushing in points from infinity, for example by the map

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 1 + \max |z_i|).$$

For any open manifold M, similar maps

$$C_n(M) \mapsto C_{n+1}(M)$$

exist by pushing in a point from the boundary. McDuff [McD75] and Segal [Seg79] proved that $H^i(C_n(M), \mathbb{Z})$ stabilizes for $n \gg i$.

For closed manifolds M, there is no direct way to compare $C_n(M)$ and $C_{n+1}(M)$. With rational coefficients however, the transfer isomorphism

$$H^*(C_n(M), \mathbb{Q}) = H^*(F_n(M), \mathbb{Q})^{S_n}$$

allows to compute $H^*(C_n(M), \mathbb{Q})$ if we understand the S_n-representation theory of $H^*(F_n(M), \mathbb{Q})$. For example, we can read off the multiplicity of the trivial representation from the description of the action of the symmetric group on $H^*(F_n(\mathbb{C}), \mathbb{Q})$ by [CT93], [LS86]. We get (compare to theorem 3.1):

$$H^0(C_n(\mathbb{C}), \mathbb{Q}) = \mathbb{Q} \qquad H^1(C_n(\mathbb{C}), \mathbb{Q}) = \mathbb{Q} \text{ if } n \geq 2 \qquad H^i(C_n(\mathbb{C}), \mathbb{Q}) = 0 \text{ if } i \geq 2$$

For ordered configuration spaces, via the maps $F_{n+1}(M) \to F_n(M)$ we can compare S_n-representations on $H^*(F_n(M), \mathbb{Q})$ and S_{n+1}-representations on $H^*(F_{n+1}(M), \mathbb{Q})$. Farb and Church found the appropriate framework of *representation stability* [CF13]. Take any integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$. For $n \gg 0$ this defines a partition $(n - \sum \lambda_i, \lambda_1, \ldots, \lambda_l)$ of n. We write $V(\lambda)_n$ for the corresponding representation of S_n. For example $V(0)$ is the trivial representation and $V(1)$ the standard representation.

Theorem 3.2. *[Chu12] (Representation Stability) Let M be a connected orientable manifold M of finite type. Then for any partition λ, the multiplicity of $V(\lambda)_n$ in $H^i(C_n(M), \mathbb{Q})$ stabilizes for $n \gg i$.*

Corollary 3.3. *The cohomology with rational coefficients of the unordered configuration spaces $H^i(C_n(M), \mathbb{Q})$ stabilizes for $n \gg i$.*

Representation stability is a very active field of research. With the language of FI-modules, it applies to many different sequences of S_n-representations (see for example [CEF15], [Chu+14]). However, computing the stable multiplicities of $V(\lambda)$ is hard and has been done only in few cases.

On the computational side of homological stability, Félix and Thomas constructed a differential graded $\mathbb{Q}$-Algebra $\Omega_n(M)$ depending only on the cohomology algebra $H^*(M, \mathbb{Q})$. For even-dimensional nilpotent orientable closed manifolds via rational homotopy theory [FT00] and for complex projective manifolds via algebraic simplifications of the S_n-invariants of Totaros spectral sequence [FT05], they could show that there is an isomorphism of groups

$$H^*(\Omega_n(M)) \simeq H^*(C_n(M), \mathbb{Q}).$$

From a computational point of view, their algebra is much more manageable than Totaro's - especially for $n \gg 0$. The stability is encoded in the algebra itself. In chapter 2, we will use this algebra to determine $H^*(C_n(\Sigma_1), \mathbb{Q})$ for an elliptic curve Σ_1.

Their analysis of the Totaro spectral sequence also allows Félix and Thomas to conclude:

Theorem 3.4. *[FT05] Let $K = \mathbb{Q}$ or $K = \mathbb{Z}/p\mathbb{Z}$ with $p > n$. Then for an odd-dimensional compact manifold M, the rational cohomology algebra of $C_n(M)$ with coefficients in K is isomorphic to the free graded algebra $\Lambda^n(H^*(M))$.*

With the framework of factorization homology [Knu14], Drummond-Cole and Knudsen could find a generalization of the algebra by Félix and Thomas that works for arbitrary manifolds. This allowed them to compute the cohomology of unordered configuration spaces of closed and open, oriented and unoriented surfaces. For example:

Theorem 3.5. *[DK16] There are polynomials p_g and q_g of degree $2g-1$ with rational coefficients such that*

$$\lim_{n\to\infty} \operatorname{rk} H^i(C_n(\Sigma_g), \mathbb{Z}) = p_g(i)$$

for $i \geq 5$ odd and

$$\lim_{n\to\infty} \operatorname{rk} H^i(C_n(\Sigma_g), \mathbb{Z}) = q_g(i)$$

for $i \geq 6$ even.

They also provide explicit, but rather complicated formulas for p_g and q_p. The properties and the dependence on g of these polynomials remain rather mysterious.

For integer coefficients, the situation gets more complicated, as $H^*(C_n(M), \mathbb{Z})$ is not necessarily the S_n-invariant part of $H^*(F_n(M), \mathbb{Z})$. Homological stability is no longer true in general. One example is

$$H_1(C_n(S^2), \mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z}$$

coming from the description of the fundamental group of $C_n(S^2)$ in [BC74]. In chapter 3, we will study $H^*(C_n(S^2), \mathbb{Z})$ by an explicit cell complex.

4. Virtual Poincaré Polynomials

As we have seen, computing the Betti numbers of configuration spaces is rather involved. One idea to attack a more manageable problem is to study the *virtual*

Poincaré polynomials of configuration spaces: For any complex quasi-projective variety X, there is a polynomial $S(X) \in \mathbb{Z}[x]$ characterized by the following properties:

- If X is smooth and projective, then $S(X)$ agrees with the usual Poincaré polynomial
$$S(X) = \sum \operatorname{rk} H^i(X, \mathbb{Q})\, x^i.$$
- If Z is a closed subvariety of X, then $S(X) = S(X \setminus Z) + S(Z)$.
- It satisfies the Künneth formula $S(X \times Y) = S(X)S(Y)$.

As the configuration space $F_n(X)$ is the complement of the diagonals in X^n, its virtual Poincaré polynomials is much easier to compute. Getzler [Get95], [Get99] could provide a complete description of the virtual Poincaré polynomials of configuration spaces of smooth projective varieties.

Theorem 4.1. *[Get95] Let X be a quasi-projective variety with virtual Poincaré polynomial $S(X) = \sum_i s_i x^i$. Then*
$$S(F_n(X)) = S(X)(S(X) - 1) \cdots (S(X) - (n-1))$$
and
$$\sum_{n=0}^{\infty} S(C_n(X)) y^n = \prod_{i=0}^{\infty} \left(\frac{1 - y^2 x^i}{1 - y x^i} \right)^{s_i}.$$

He could even give S_n-equivariant version of these formulas. As an example, we compute the ordinary and virtual Poincaré polynomials of configuration spaces of $\mathbb{C} \setminus k$ in chapter 5.

Let E be an elliptic curve E with neutral element 0. In chapter 4 we show that a variant of Getzlers approach can be used of to determine the virtual Poincaré polynomials of the space
$$F_n^0(E) = \{(x_1, \ldots, x_n) \in E^n |\, x_i \neq x_j \text{ and } \sum x_i = 0\}$$
of n distinct points on E with sum 0

CHAPTER 2

Betti numbers of unordered Configuration spaces of the Torus

1. Introduction

In the context of representation stability, Church showed that for a connected, orientable manifold M of finite type the rational cohomology groups $H^i(C_n(M), \mathbb{Q})$ stabilise for $n > i$ [Chu12, Cor. 3]. However, very few of these stable Betti numbers have been explicitly computed. Félix and Thomas [FT00] showed that for a closed, orientable, nilpotent, even-dimensional manifold M, the rational Betti numbers of $C_n(M)$ are determined by the rational cohomology algebra $H^*(M, \mathbb{Q})$. They constructed an explicit differential graded algebra that we use to compute the Betti numbers of the unordered configuration spaces of the torus $\Sigma_1 = S^1 \times S^1$. These numbers were previously unknown.

Theorem 1.1. *Suppose $n \geq 2$. Then*

$$\dim_{\mathbb{Q}} H^i(C_n(\Sigma_1), \mathbb{Q}) = \begin{cases} \frac{n-2}{2} & i = n+1,\ n \text{ even} \\ \frac{n+1}{2} & i = n+1,\ n \text{ odd} \\ \frac{3n-4}{2} & i = n,\ n \text{ even} \\ \frac{3n-1}{2} & i = n,\ n \text{ odd} \\ 2i-1 & 2 \leq i < n \\ 2 & i = 1 \\ 1 & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Azam [Aza15] determined the rational Betti numbers of configuration spaces of Riemann surfaces for $n = 2, 3$ in any genus and for $n = 4$ in genus 1 by the Kriz model [Kri94]. Napolitano [Nap03] computed the integral cohomology groups of $C_n(\Sigma_1)$ for $n \leq 7$ using a cellular decomposition. Indepent of our work, the Betti numbers of unordered configuration spaces were computed for the torus by Maguire and for surfaces of any genus by Drummond-Cole and Knudsen using more sophisticated, but more general approaches [MCF16] [DK16].

The theorem has been tested for all $n \leq 20$ using the computer algebra system SAGE [Sage].

2. Conventions

We consider $n \geq 2$ as $C_1(X) \simeq X$. In this chapter, we will always work with cohomology/homology with $\mathbb{Q}$-coefficients and identify

$$H^*(M, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(H_*(M, \mathbb{Q}), \mathbb{Q}).$$

The free $\mathbb{Q}$-vector space with basis $x_1, \ldots, x_n$ is denoted by $\langle x_1, \ldots, x_n \rangle$.

For any differential graded commutative algebra (A, d), we use the sign convention $xy = (-1)^{\deg x \deg y} yx$ and $d(xy) = d(x)y + (-1)^{\deg x} x d(y)$ for homogenous $x, y \in A$. We have the free graded commutative algebra $\Lambda(V)$ on any graded vector space V with

$$\Lambda(V) = \text{Exterior algebra } (V^{\mathrm{odd}}) \otimes \text{Symmetric algebra } (V^{\mathrm{even}}).$$

3. Construction of the Algebra

Let M be a manifold. The cup product gives a map

$$\cup \colon H^*(M) \otimes H^*(M) \to H^*(M),$$

which dualizes to the diagonal comultiplication

$$\Delta \colon H_*(M) \to H_*(M) \otimes H_*(M).$$

Using a basis e_i of $H^*(M)$ the map Δ is given by

$$\Delta(e_k^*) = \sum_{i,j} (\text{coefficient of } e_k \text{ in } e_i \cup e_j)\ e_i^* \otimes e_j^*,$$

where e_i^* denotes the dual basis of $H_*(M)$.

Set $m = \dim(M)$. We take two shifted copies V, W of the vector space $H_*(M)$ with (upper) grading

$$V^{m-r} = H_r(M) \qquad\qquad W^{2m-1-r} = H_r(M).$$

We endow the free graded algebra $\Omega = \Lambda(V \oplus W)$ with the unique differential D of degree 1 such that

$$D_{|V} = 0 \qquad\qquad D_{|W} \colon W \simeq H_*(M) \xrightarrow{\Delta} \Lambda^2 H_*(M) \simeq \Lambda^2 V.$$

A lower grading

$$\Omega = \bigoplus_{n \geq 0} \Omega_n$$

can be defined by putting V in degree 1 and W in degree 2. Hence we have

$$\Omega_n = \bigoplus_{r+2s=n} \Lambda^r V \otimes \Lambda^s W.$$

The vectorspace Ω_n is also graded

$$\Omega_n = \bigoplus_{i \geq 0} \Omega_n^i$$

by the upper grading inherited from Ω. As $D(W) \subset \Lambda^2 V$, the differential D respects the lower grading and Ω_n is a subcomplex of (Ω, D).

Félix and Thomas showed that the algebra (Ω_n, D) is a model for the cohomology of $H^*(C_n(M), \mathbb{Q})$.

Theorem 3.1. *[FT00, Th. A(2)] Let M be an orientable, closed, nilpotent, even-dimensional manifold. There is an isomorphism of graded vector spaces*

$$H^*(C_n(M), \mathbb{Q}) \simeq H^*(\Omega_n, D).$$

4. Configuration Spaces of the Torus

Now we apply this theorem for the torus Σ_1. Its cohomology algebra is $H^*(\Sigma_1) = \langle 1, a, b, ab\rangle$ with $\deg(a) = \deg(b) = 1$ and the relations $ab = -ba$, $a^2 = b^2 = 0$. As $\pi_1(\Sigma_1) = \mathbb{Z}^2$ is abelian and the higher homotopy groups of Σ_1 vanish, Σ_1 is a nilpotent space. We introduce the graded vector spaces $V = \langle v_1, v_a, v_b, v_{ab}\rangle$ and $W = \langle w_1, w_a, w_b, w_{ab}\rangle$ with degrees

$$\begin{aligned} \deg v_1 &= 2 & \qquad \deg w_1 &= 3 \\ \deg v_a &= 1 & \deg w_a &= 2 \\ \deg v_b &= 1 & \deg w_b &= 2 \\ \deg v_{ab} &= 0 & \deg w_{ab} &= 1. \end{aligned}$$

We look at the graded algebra $\Omega = \Lambda\langle v_1, v_a, v_b, v_{ab}, w_1, w_a, w_b, w_{ab}\rangle$ with the differential D given by

$$\begin{aligned} D(v_1) &= 0 & \qquad D(w_1) &= v_1^2 \\ D(v_a) &= 0 & D(w_a) &= 2v_1v_a \\ D(v_b) &= 0 & D(w_b) &= 2v_1v_b \\ D(v_{ab}) &= 0 & D(w_{ab}) &= 2v_1v_{ab} + 2v_av_b. \end{aligned}$$

By Theorem 3.1 we have to compute the cohomology groups of the subcomplexes

$$\Omega_n = \bigoplus_{r+2s=n} \Lambda^r V \oplus \Lambda^s W.$$

We will do this by embedding them into the algebra

$$\Theta = \Lambda\langle v_1, v_a, v_b, w_1, w_a, w_b, w_{ab}\rangle,$$

with differential d given by:

$$\begin{aligned} d(v_1) &= 0 & \qquad d(w_1) &= v_1^2 \\ d(v_a) &= 0 & d(w_a) &= 2v_1v_a \\ d(v_b) &= 0 & d(w_b) &= 2v_1v_b \\ & & d(w_{ab}) &= 2v_1 + 2v_av_b. \end{aligned}$$

All variables have the same grading as in Ω; we only set $v_{ab} = 1$.

Lemma 4.1. *There is an isomorphism $H^i(\Omega_n, D) \simeq H^i(\Theta, d)$ for $i < n$.*

Proof. The injective map

$$\pi \colon \Omega_n \to \Theta, \quad v_{ab} \mapsto 1$$

respects the grading as $\deg v_{ab} = 0$ and commutes with the differentials. Take a degree $i < n$ and any monomial

$$\prod v_k^{e_k} \prod w_l^{f_l} \in \Theta^i$$

of degree i. The only generators of Ω where the lower degree exceeds the upper one are v_{ab} and w_{ab}. As $w_{ab}^2 = 0$ we see

$$\sum e_k + 2\sum f_l \leq i + 1.$$

So the monomial

$$v_{ab}^{n-\sum e_k - 2\sum f_l} \prod v_k^{e_k} \prod w_l^{f_l}$$

is in Ω_n and

$$\pi(v_{ab}^{n-\sum e_k - 2\sum f_l} \prod v_k^{e_k} \prod w_l^{f_l}) = \prod v_k^{e_k} \prod w_l^{f_l}.$$

Thus π is also surjective in degree i. Altogether, π induces an isomorphism

$$H^i(\Omega_n, d) \simeq H^i(\Theta, d)$$

for $i < n$. □

In order to compute the Betti numbers of (Θ, d), we compare d with the simpler differential d_0 given by

$$\begin{aligned} d_0(v_1) &= 0 & d_0(w_1) &= 0 \\ d_0(v_a) &= 0 & d_0(w_a) &= 0 \\ d_0(v_b) &= 0 & d_0(w_b) &= 0 \\ & & d_0(w_{ab}) &= 2v_1 + 2v_a v_b \end{aligned}$$

Lemma 4.2. *There is an isomorphism $\varphi\colon (\Theta, d_0) \to (\Theta, d)$.*

Proof. It can be explicitly given by

$$\begin{aligned} \varphi(v_1) &= v_1 & \varphi(w_1) &= w_1 - \frac{1}{2}v_1 w_{ab} + \frac{1}{2}v_b w_a \\ \varphi(v_a) &= v_a & \varphi(w_a) &= w_a + v_a w_{ab} \\ \varphi(v_b) &= v_b & \varphi(w_b) &= w_b + v_b w_{ab} \\ & & \varphi(w_{ab}) &= w_{ab} \end{aligned}$$

As $d(\varphi(w_1)) = d(\varphi(w_a)) = d(\varphi(w_b)) = 0$, the map φ commutes with the differentials. □

Lemma 4.3. *The Betti numbers of $H^*(\Theta, d_0)$ are*

$$\dim_{\mathbb{Q}} H^i(\Theta, d_0) = \begin{cases} 1 & i = 0 \\ 2 & i = 1 \\ 2i - 1 & i \geq 2. \end{cases}$$

Proof. Denote $T = \Lambda\langle v_1, v_a, v_b, w_1, w_a, w_b\rangle$. Then we have $\Theta = T \oplus T w_{ab}$. Observe that $d_0|T = 0$. Take homogenous $x, y \in T$ and compute

$$d_0(x + y w_{ab}) = d_0(x) + d_0(y) w_{ab} \pm y d_0(w_{ab}) = \pm 2y(v_1 + v_a v_b).$$

As v_1 has even degree, $v_1 + v_a v_b$ is not a zero-divisor. So we know that $\mathrm{Ker}(d_0) = T$ and

$$H^*(\Theta, d_0) = T/(v_1 + v_a v_b) \simeq T/(v_1) \simeq \Lambda\langle v_a, v_b, w_1, w_a, w_b\rangle.$$

The Poincaré series of $\Lambda\langle v_a, v_b, w_1, w_a, w_b\rangle$ is

$$\frac{(1 + t^{\deg v_a})(1 + t^{\deg v_b})(1 + t^{\deg w_1})}{(1 - t^{\deg w_a})(1 - t^{\deg w_b})} = \frac{(1+t)^2(1+t^3)}{(1-t^2)^2} = \frac{1+t^3}{(1-t)^2},$$

which expands to

$$1 + 2t + 3t^2 + 5t^3 + 7t^4 + \cdots + (2i-1)t^i + \cdots. \qquad \square$$

Combining Lemmas 4.1, 4.2 and 4.3 we have computed $\dim_{\mathbb{Q}} H^i(\Omega_n)$ for $i < n$.

Remark 4.4. We consider the morphism

$$p\colon \Omega_n \to \Lambda\langle v_a, v_b, w_1, w_a, w_b, w_{ab}\rangle,\ v_{ab} \mapsto 1,\ v_1 \mapsto -v_a v_b.$$

The above proof shows that for any $x \in \mathrm{Im}\, D$ necessarily $p(x) = 0$.

Lemma 4.5. *We have*

$$\dim_{\mathbb{Q}} H^{n+1}(\Omega_n) = \begin{cases} \frac{n-2}{2} & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{cases} \qquad \dim_{\mathbb{Q}} H^i(\Omega_n) = 0 \text{ for } i > n+1.$$

Proof. We denote $\Theta' = \Lambda\langle v_1, v_a, v_b, v_{ab}, w_a, w_b, w_{ab}\rangle$. The only generators with upper grading exceeding the lower grading are v_1 and w_1. Hence any $x \in \Omega_n^i$ with $i > n$ can be written as $x = v_1 f + w_1 g$ where $f, g \in \Theta'$. We compute

$$D(x) = v_1 D(f) + v_1^2 g - w_1 D(g).$$

As $D(\Theta') \subset \Theta'$ we see that $D(x) = 0$ implies $D(g) = 0$. So $x \in \mathrm{Ker}\, D$ if and only if $D(f) = -v_1 g$. Therefore any $x \in \mathrm{Ker}\, D$ is of the form

$$x(f) = v_1 f - w_1 \frac{D(f)}{v_1}$$

with $f \in \Theta'$ such that $v_1 | D(f)$.

We will now discuss when the cycles $x(f)$ are a boundary. If $f = v_1 h$, then

$$D(w_1 h) = v_1^2 h - w_1 D(h) = v_1 f - w_1 \frac{D(f)}{v_1} = x(f).$$

For any $x(f) \in \Omega_n^i$ with $i > n+1$ we know that f has to be divisible by v_1. Hence $H^i(\Omega_n) = 0$ for $i > n+1$.

Now we look at the case $i = n+1$. We consider the sets

$$B_{\mathrm{odd}} = \{w_a^{n_1} w_b^{n_2} \mid 2n_1 + 2n_2 + 1 = n;\ n_1, n_2 \geq 0\}$$

for odd degree n and

$$B_{\mathrm{even}} = \{v_b w_a w_a^{n_1} w_b^{n_2} \mid 2n_1 + 2n_2 + 4 = n;\ n_1, n_2 \geq 0\}$$

for even n.

If $v_{ab}|f$ or $w_{ab}|f$, then $v_1|f$ for degree reasons. If $v_1 f$ is a boundary $v_1 f = D(h)$, then $D(f) = 0$ and hence $x(f) = D(h) \in \mathrm{Im}(D)$. So using the relations

$$\begin{aligned} D(v_b w_a^{n_1+1} w_b^{n_2}) &= -2(n_1+1) v_1 v_a v_b w_a^{n_1} w_b^{n_2} \\ D(w_a^{n_1+1}) &= 2(n_1+1) v_1 v_a w_a^{n_1} \\ D(w_b^{n_1+1}) &= 2(n_1+1) v_1 v_b w_b^{n_1} \\ D(w_a^{n_1+1} w_b^{n_2+1}) &= 2(n_1+1) v_1 v_a w_a^{n_1} w_b^{n_2+1} + 2(n_2+1) v_1 v_b w_a^{n_1+1} w_b^{n_2} \end{aligned}$$

we conclude that the set $\{x(b) | b \in B_{\text{even}}\}$ resp. $\{x(b) | b \in B_{\text{odd}}\}$ is a generating system of $H^{n+1}(\Omega_n)$ for even resp. odd n.

By applying p, we see that no non-trivial linear combinations of these generating sets are boundaries. Hence we found an explicit basis of $H^{n+1}(\Omega_n)$. □

Lemma 4.6. *We have*

$$\dim_{\mathbb{Q}} H^n(\Omega_n) = \begin{cases} \frac{3n-4}{2} & n \text{ even} \\ \frac{3n-1}{2} & n \text{ odd} \end{cases}.$$

Proof. As the torus acts freely on $C_n(\Sigma_1)$, we have $\chi(\Omega_n) = 0$. Using the above computation of $\dim_{\mathbb{Q}} H^{n+1}(\Omega_n)$ and

$$\sum_{i=0}^{n-1} \dim_{\mathbb{Q}} H^i(\Omega_n) = 1 - 2 + 3 + \cdots + (-1)^{n-1}(2n-3) = (-1)^{n-1}(n-1)$$

we can reconstruct the only missing Betti number $\dim_{\mathbb{Q}} H^n(\Omega_n)$. □

Combining all lemmas, we have computed $\dim_{\mathbb{Q}} H^i(C_n(\Sigma_1), \mathbb{Q})$ for all i. We reproduce exactly the stability result

$$\dim_{\mathbb{Q}} H^i(C_{n+1}(\Sigma_1), \mathbb{Q}) = \dim_{\mathbb{Q}} H^i(C_n(\Sigma_1), \mathbb{Q})$$

for $n > i$ of Church [Chu12, Cor. 3].

Remark 4.7. Let $d \geq 1$. With the same method one immediately finds for $n \geq 3$

$$\dim_{\mathbb{Q}} H^i(C_n(S^{2d}), \mathbb{Q}) = \begin{cases} 1 & \text{for } i = 0,\, 4d-1 \\ 0 & \text{otherwise,} \end{cases}$$

which has also been computed by [Ran13], [Sal04].

Remark 4.8. It seems that our method does not work for surfaces of genus $g > 1$ because the differential can not be deformed as in Lemma 4.2.

CHAPTER 3

Integral Cohomology of Configuration Spaces of the Sphere

We compute the cohomology of the unordered configuration spaces of the sphere S^2 with integral and with $\mathbb{Z}/p\mathbb{Z}$-coefficients using a cell complex by Fuks, Vainshtein and Napolitano.

1. Introduction

1.1. Representation Stability. Arnold [Arn70] showed that all of the cohomology groups of $C_n(\mathbb{C})$ are finite, except

$$H^0(C_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \qquad\qquad H^1(C_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \text{ if } n \geq 2$$

and stabilize:

$$H^r(C_n(\mathbb{C}), \mathbb{Z}) = H^r(C_{n+1}(\mathbb{C}), \mathbb{Z}) \text{ if } n \geq 2r - 2.$$

For rational coefficients, Church [Chu12, Cor. 3] could prove that

$$H^r(C_n(M), \mathbb{Q}) = H^r(C_{n+1}(M), \mathbb{Q}) \text{ if } n > r + 1$$

for any connected, orientable manifold M of finite type. This is called *homological stability*. One example is ([Sev84], [Ran13], [Sal04]):

$$H^r(C_n(S^2), \mathbb{Q}) = \begin{cases} \mathbb{Q} & n \geq 3, r = 3 \\ \mathbb{Q} & n = 1, r = 2 \\ \mathbb{Q} & r = 0 \\ 0 & \text{otherwise} \end{cases}$$

With integer coefficients however, homological stability turns out to be false in general. For example the computation of $\pi_1 C_n(S^2)$ in [BC74, Th. 1.11] shows that:

$$H_1(C_n(S^2), \mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z}.$$

With $\mathbb{Z}/p\mathbb{Z}$-coefficients, homological stability can be replaced by *eventual periodicity*

$$H^r(C_n(M), \mathbb{Z}/p\mathbb{Z}) = H^r(C_{n+p}(M), \mathbb{Z}/p\mathbb{Z}) \text{ if } n > 2r$$

for any connected manifold M of finite type [Nag15], [CP15], [KM16].

In this chapter, we will give an example of this phenomenon by computing the cohomology groups of $C_n(S^2)$ using a cellular complex.

1.2. Cohomology of $C_n(\mathbb{C})$. Let p be a prime. Then Fuks [Fuk70] (for $p = 2$) and Vainshtein [Vai78] gave a combinatorial formula for the groups $H^r(C_n(\mathbb{C}), \mathbb{Z}_p)$. Define

$$B_p(n,r) = \left| \left\{ \begin{matrix} 1 \leq a_1 \leq a_2 \leq \cdots \leq a_g \\ 0 \leq b_1 < b_2 < \cdots < b_h \end{matrix} \middle| \begin{matrix} 2\sum_i p^{a_i} + 2\sum_j p^{b_j} - 2g - h = r \\ 2\sum_i p^{a_i} + 2\sum_j p^{b_j} \leq n \end{matrix} \right\} \right|.$$

They could show that

$$\dim H^r(C_n(\mathbb{C}), \mathbb{Z}/p\mathbb{Z}) = B_p(n,r).$$

1.3. Cohomology of $C_n(S^2)$. Using a cellular decomposition of $C_n(S^2)$ by Napolitano [Nap03], we compute the cohomology groups of $C_n(S^2)$ with $\mathbb{Z}/p\mathbb{Z}$-coeffcients in this paper.

Theorem 1.1. *Let*

$$B'_p(n,r) = \left| \left\{ \begin{matrix} 1 \leq a_1 \leq a_2 \leq \cdots \leq a_g \\ 1 \leq b_1 < b_2 < \cdots < b_h \end{matrix} \middle| \begin{matrix} 2\sum_i p^{a_i} + 2\sum_j p^{b_j} + 1 - 2g - h = r \\ 2\sum_i p^{a_i} + 2\sum_j p^{b_j} + 2 \leq n \\ p \nmid 2(n - 2\sum_i p^{a_i} - 2\sum_j p^{b_j} - 1) \end{matrix} \right\} \right|.$$

Then

$$\dim H^r(C_n(S^2), \mathbb{Z}/p\mathbb{Z}) = B_p(n,r) + B_p(n-1, r-2) - B'_p(n,r) - B'_p(n, r-1).$$

Corollary 1.2. *We have*

$$\dim H^r(C_n(S^2), \mathbb{Z}/2\mathbb{Z}) = B_2(n,r) + B_2(n-1, r-2).$$

Eventual periodicity of $H^r(C_n(S^2), \mathbb{Z}/p\mathbb{Z})$ can be directly concluded from this description. Theorem 1.1 could also be deduced from [Sal04, Th. 18.3]. However, our approach is more elementary and allows to determine the integral cohomology:

Theorem 1.3. *The first cohomology groups $H^r(C_n(S_2), \mathbb{Z})$ are*

$$H^0(C_n(S_2), \mathbb{Z}) = \mathbb{Z} \qquad H^1(C_n(S_2), \mathbb{Z}) = 0$$

$$H^2(C_n(S_2), \mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z} \qquad H^3(C_n(S_2), \mathbb{Z}) = \begin{cases} 0 & n = 1,2 \\ \mathbb{Z} & n = 3 \\ \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & n \geq 4 \end{cases}$$

For $r \geq 4$, the cohomology groups $H^r(C_n(S^2), \mathbb{Z})$ are finite and contain no elements of order p^2.

Hence we can reconstruct all integral cohomology groups by theorem 1.1 and the universal coefficient theorem. The description of $H^r(C_n(S^2), \mathbb{Z})$ seems to be new.

We will first explain the computations of the cohomology of $C_n(\mathbb{C})$ with $\mathbb{Z}/p\mathbb{Z}$-coefficients by Fuks [Fuk70] and Vainshtein [Vai78] and discuss their cell complex. Afterwards, we present the extension of this cell complex that Napolitano [Nap03] used to calculate $H^*(C_n(S^2), \mathbb{Z})$ for $n \leq 9$. The main idea of this paper is the construction of a chain homotopy that simplifies Napolitano's complex.

2. Configuration Spaces of the Plane

2.1. Conventions. We write

$$\mathrm{Comb}(n,q) = \{\, [n_1,\dots,n_q] \in \mathbb{Z}^q_{>0} \,|\, n_1 + \cdots + n_q = n\}$$

for compositions of n into q positive summands, for example

$$\mathrm{Comb}(5,3) = \{[3,1,1],\ [1,3,1],\ [1,1,3],\ [2,2,1],\ [2,1,2],\ [1,2,2]\,\}.$$

We call q the length and n the size of the composition.

The residue ring $\mathbb{Z}/m\mathbb{Z}$ is denoted $\mathbb{Z}_m$. For any abelian group G and prime p, we write $G_{(p)} = \{g \in G |\, p^n g = 0 \text{ for some } n\}$ for the p-torsion subgroup.

2.2. Cellular Decomposition of $\overline{C_n(\mathbb{C})}$. The following construction comes from [Fuk70] and [Vai78]: The projection

$$\mathbb{C} \to \mathbb{R}, x + iy \mapsto x$$

to the real line maps any configuration in $C_n(\mathbb{C})$ to a finite sets of points in $\mathbb{R}$. Counting the number of preimages of each of these points, we get a composition of n. The union of all points in $C_n(\mathbb{C})$ mapping to the same composition $n = n_1 + \cdots + n_s$ and the point ∞ is a $n + s$-dimensional cell in the one point compactification $\overline{C_n(\mathbb{C})}$. We denote this cell by $[n_1,\dots,n_s]$. All such cells together with the point ∞ are a cellular decomposition of $\overline{C_n(\mathbb{C})}$. Using Poincaré-Lefschetz duality for Borel-Moore homology [CG10] [Vas01]

$$H^i(C_n(\mathbb{C})) = \tilde{H}_{2n-i}(\overline{C_n(\mathbb{C})}),$$

this cell complex can be used to compute the cohomology of $C_n(\mathbb{C})$.

The (co)-chains of the resulting (co)-complex $A_n^\bullet = (A_n^r)_r$ with the property

$$H^*(C_n(\mathbb{C}), \mathbb{Z}) = H^*(A_n^\bullet)$$

are the free $\mathbb{Z}$-modules

$$A_n^r = \mathbb{Z}\,\mathrm{Comb}(n, n-r).$$

The basis elements are the compositions $[n_1,\dots,n_s] \in \mathrm{Comb}(n,s)$ with $s = n - r$. The boundary maps $\delta\colon A_n^r \to A_n^{r+1}$ are

$$\delta[n_1,\dots,n_s] = \sum_{l=1}^{s-1} (-1)^{l-1} P(n_l, n_{l+1})[n_1,\dots,n_{l-1}, n_l + n_{l+1}, n_{l+2},\dots,n_s]$$

where

$$P(x,y) = \begin{cases} 0 & \text{if } x \equiv y \equiv 1 \mod 2, \\ \dbinom{\lfloor x/2 + y/2 \rfloor}{\lfloor x/2 \rfloor} & \text{otherwise.} \end{cases}$$

2.3. Cohomology of $C_n(\mathbb{C})$. As $P(x,y)=0$ for odd x and y, the complex $A_n^\bullet$ can be written as a direct sum

$$A_n^\bullet = A_{n,0}^\bullet \oplus \cdots \oplus A_{n,n}^\bullet$$

of subcomplexes $A_{n,t}^\bullet$ generated by compositions with t odd entries.

Take any $I \subset \{1,\dots,s+t\}$ with t elements, say $I=\{i_1,\dots,i_t\}$ where $i_1<\cdots<i_t$. Then we insert $1's$ at the positions i_1 to i_t with alternating signs:

$$\mathrm{Ins}_I[a_1,\dots,a_s] = (-1)^{\sum_j i_j}[a_1,\dots,a_{i_1-1},1,a_{i_1},\dots,a_{i_2-2},1,a_{i_2-1},\dots]$$

The map

$$\mathrm{Ins}_t = (-1)^{st}\sum_{I\subset\{1,\dots,s+t\},|I|=t} \mathrm{Ins}_I$$

is actually a chain map

$$\mathrm{Ins}_t\colon A_{n,0}^\bullet \to A_{n+t,t}^\bullet$$

that induces isomorphisms

$$H^r(A_{n-t,0}^\bullet) \simeq H^r(A_{n,t}^\bullet).$$

Hence we get

$$H^*(A_n^\bullet) = H^*(A_{n,0}^\bullet) \oplus H^*(A_{n-1,0}) \oplus \cdots \oplus H^*(A_{0,0}^\bullet).$$

As $A_{n,0}^r = 0$ if $n\equiv 1 \mod 2$ or $n>2r$, we can immediately deduce the properties of recurrence and stability of theorem 3.1 in chapter 1. We write

$$H^r(C_\infty(\mathbb{C})) = \lim_{n\to\infty} H^r(C_n(\mathbb{C})).$$

EXAMPLE 2.1. The cohomology group $H^0(C_n(\mathbb{C}),\mathbb{Z})=\mathbb{Z}$ is generated by the class of $(-1)^{n(n-1)/2}[1,\dots,1]=\mathrm{Ins}_n([\,])$. For $n\geq 2$, the cohomology group $H^1(C_n(\mathbb{C}),\mathbb{Z})=\mathbb{Z}$ is generated by the class of $[2,1,\dots,1]-[1,2,1,\dots,1]+\cdots = (-1)^{(n-2)(n-3)/2+n}\,\mathrm{Ins}_{n-2}[2]$.

2.4. Explicit Basis of $H^*(A_{n,0}^\bullet,\mathbb{Z}_p)$. We will now present the description of the group $H^r(A_{n,0}^\bullet,\mathbb{Z}_p)$ by Vainshtein and work out some of the details and proofs omitted in [Vai78]. In particular, the explicit formula for the base elements is misleading and seems to be wrong in the stated form in [Vai78].

Let $[n_1,\dots,n_s]$ be any composition of n. Then the alternating sum of its permutations

$$\sum_{\sigma\in S_s} \mathrm{sign}(\sigma)[n_{\sigma(1)},\dots,n_{\sigma(s)}]$$

is a cycle in $A_n^\bullet$. With $\mathbb{Z}_p$-coefficients, the following subset of permutations

$$\mathrm{Perm}[n_1,\dots,n_s] = \sum_{\substack{\sigma\in S_s \text{ where } \sigma(i)<\sigma(j)\\ \text{if } i<j \text{ and } n_i=n_j \text{ or}\\ \text{if } i<j \text{ and } P(n_i,n_j)=0 \bmod p}} \mathrm{sign}(\sigma)[n_{\sigma(1)},\dots,n_{\sigma(s)}]$$

creates a cycle in $A_n^\bullet\otimes\mathbb{Z}_p$.

Take integers $1\leq i_1\leq\cdots\leq i_k$ and $0\leq j_1<\cdots<j_l$ such that

$$m = n-2(p^{i_1}+\cdots+p^{i_k}+p^{j_1}+\cdots+p^{j_l})\geq 0$$

and let

$$r = (2p^{i_1}-2)+\cdots+(2p^{i_k}-2)+(2p^{j_1}-1)+\cdots+(2p^{j_l}-1).$$

Then we give the chain

$$\mathrm{Ins}_m \operatorname{Perm}[2p^{i_1-1}, 2p^{i_1-1}(p-1), \ldots, 2p^{i_k-1}, 2p^{i_k-1}(p-1), 2p^{j_1}, \ldots, 2p^{j_l}]$$

the name $x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_l}$. It is a cycle in $A^r_{n,m} \otimes \mathbb{Z}_p$ (but not in $A^\bullet_n$ if $k > 0$). Vainshtein showed that all such cycles form a basis of $H^r(A^\bullet_n, \mathbb{Z}_p)$. We call the quantity $n - m$ the size of the chain $x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_l}$.

Theorem 2.2. *[Vai78] [CLM07] The ring $H^*(C_\infty, \mathbb{Z}_p)$ is the free graded commutative algebra over $\mathbb{Z}_p$ with generators*

$$\begin{array}{lll} x_i \text{ for } i \geq 1 & \deg(x_i) = 2p^i - 2 & \operatorname{size}(x_i) = 2p^i \\ y_i \text{ for } i \geq 0 & \deg(y_i) = 2p^i - 1 & \operatorname{size}(y_i) = 2p^i. \end{array}$$

There is a surjection $H^*(C_\infty(\mathbb{C}), \mathbb{Z}_p) \to H^*(C_n(\mathbb{C}), \mathbb{Z}_p)$ whose kernel is generated by the monomials $x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_l}$ such that $\operatorname{size}(x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_l}) > n$.

Corollary 2.3. *Define*

$$B_p(n,r) = \left| \left\{ \begin{matrix} 1 \leq a_1 \leq a_2 \leq \cdots \leq a_g \\ 0 \leq b_1 < b_2 < \cdots < b_h \end{matrix} \middle| \begin{matrix} 2\sum_i p^{a_i} + 2\sum_j p^{b_j} - 2g - h = r \\ 2\sum_i p^{a_i} + 2\sum_j p^{b_j} \leq n \end{matrix} \right\} \right|.$$

Hence we have

$$\dim H^r(C_n(\mathbb{C}), \mathbb{Z}_p) = B_p(n,r).$$

Corollary 2.4. *[Sal04] This can also be written as a generating series:*

$$\sum_{n,r \geq 0} B_p(n,r) w^r z^n = \frac{1 + wz^2}{1 - z} \prod_{i>0} \frac{1 + w^{2p^i - 1} z^{2p^i}}{1 - w^{2p^i - 2} z^{2p^i}}$$

Remark 2.5. The notation suggests a product structure on $H^*(C_\infty(\mathbb{C}), \mathbb{Z}_p)$. It comes from the map

$$C_n(\mathbb{C}) \times C_m(\mathbb{C}) \to C_{n+m}(\mathbb{C})$$

by adding the points far apart.

Remark 2.6. As

$$\binom{p^a + p^b}{p^a} \equiv \begin{cases} 1 & a \neq b \\ 2 & a = b \end{cases} \mod p$$

and

$$\binom{p^a + p^b(p-1)}{p^a} \equiv \begin{cases} 1 & a \neq b \\ 0 & a = b \end{cases} \mod p$$

by Lucas's theorem [Fin47], the order of all entries of the form $2p^a$, $2p^a(p-1)$ in our basis elements is preserved by the operator Perm.

Example 2.7. We compute $H^*(C_{24}(\mathbb{C}), \mathbb{Z}/3\mathbb{Z})$. The generators have degrees

generators	x_1	x_2	y_0	y_1	y_2	…
degree	4	16	1	5	17	…
size	6	18	2	6	18	…

In table 1, we write down the basis elements and the corresponding chains, however we will omit the application of the Ins_t-operators to lift the chains to sum 24.

TABLE 1. The cohomology group $H^*(C_{24}(\mathbb{C}), \mathbb{Z}_3)$

r	basis of $H^r(C_{24}(\mathbb{C}), \mathbb{Z}_3)$
0	$1 = []$
1	$y_0 = [2]$
2	–
3	–
4	$x_1 = [2, 4]$
5	$y_1 = [6]$ $x_1 y_0 = [2, 4, 2]$
6	$y_0 y_1 = [2, 6] - [6, 2]$
7	–
8	$x_1^2 = [2, 4, 2, 4]$
9	$x_1 y_1 = [2, 4, 6] - [2, 6, 4] + [6, 2, 4]$ $x_1^2 y_0 = [2, 4, 2, 4, 2]$
10	$x_1 y_0 y_1 = [2, 4, 2, 6] - [2, 4, 6, 2] + [2, 6, 4, 2] - [6, 2, 4, 2]$
11	–
12	$x_1^3 = [2, 4, 2, 4, 2, 4]$
13	$x_1^2 y_1 = [2, 4, 2, 4, 6] - [2, 4, 2, 6, 4] + [2, 4, 6, 2, 4] - [2, 6, 4, 2, 4] + [6, 2, 4, 2, 4]$ $x_1^3 y_0 = [2, 4, 2, 4, 2, 4, 2]$
14	$x_1^2 y_0 y_1 = [2, 4, 2, 4, 2, 6] - [2, 4, 2, 4, 6, 2] + [2, 4, 2, 6, 4, 2] - [2, 4, 6, 2, 4, 2,] + \dots$
15	–
16	$x_2 = [6, 12]$ $x_1^4 = [2, 4, 2, 4, 2, 4, 2, 4]$
17	$y_2 = [18]$ $x_2 y_0 = [6, 12, 2] - [6, 2, 12] + [2, 6, 12]$ $x_1^3 y_1 = [2, 4, 2, 4, 2, 4, 6] - [2, 4, 2, 4, 2, 6, 4] + \dots$
18	$y_0 y_2 = [2, 18] - [18, 2]$
19	–
20	$x_1 x_2 = [2, 4, 6, 12] - [2, 6, 4, 12] + [6, 2, 4, 12] - [6, 2, 12, 4] + [2, 6, 12, 4] + [6, 12, 2, 4]$
21	$x_1 y_2 = [2, 4, 18] - [2, 18, 4] + [18, 2, 4]$ $x_2 y_1 = [6, 12, 6]$
22	$y_1 y_2 = [6, 18] - [18, 6]$
≥ 23	–

2.5. Bockstein Homomorphisms. The short exact sequences of coefficients

$$0 \to \mathbb{Z} \xrightarrow{p\cdot} \mathbb{Z} \to \mathbb{Z}_p \to 0$$

and

$$0 \to \mathbb{Z}_p \xrightarrow{p\cdot} \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$$

induce long exact sequences

$$H^{r-1}(A_n^\bullet, \mathbb{Z}_p) \xrightarrow{\tilde{\beta}} H^r(A_n^\bullet, \mathbb{Z}) \xrightarrow{p\cdot} H^r(A_n^\bullet, \mathbb{Z}) \to H^r(A_n^\bullet, \mathbb{Z}_p) \xrightarrow{\tilde{\beta}} H^{r+1}(A_n^\bullet, \mathbb{Z})$$

and

$$H^{r-1}(A_n^\bullet, \mathbb{Z}_p) \xrightarrow{\beta} H^r(A_n^\bullet, \mathbb{Z}_p) \xrightarrow{p\cdot} H^r(A_n^\bullet, \mathbb{Z}_{p^2}) \to H^r(A_n^\bullet, \mathbb{Z}_p) \xrightarrow{\beta} H^{i+1}(A_n^\bullet, \mathbb{Z}_p),$$

where the connecting morphisms are the *Bockstein morphisms* β and $\tilde{\beta}$ (compare [Hat02, Chap.3.E]). The image of $\tilde{\beta}$ are hence all elements of order p in $H^*(A_n^\bullet, \mathbb{Z})$. The following diagram commutes and the upper row is exact:

$$\begin{tikzcd}
H^r(A_n^\bullet, \mathbb{Z}) \arrow[r] & H^r(A_n^\bullet, \mathbb{Z}_p) \arrow[r, "\tilde{\beta}"] \arrow[dr, "\beta"'] & H^{r+1}(A_n^\bullet, \mathbb{Z}) \arrow[r, "p\cdot"] \arrow[d] & H^{r+1}(A_n^\bullet, \mathbb{Z}) \\
& & H^{r+1}(A_n^\bullet, \mathbb{Z}_p) &
\end{tikzcd}$$

EXAMPLE 2.8. Let $i \neq j$. We determine the Bockstein on $x_i = [2p^{i-1}, 2p^{i-1}(p-1)]$ and $x_i y_j = [2p^{i-1}, 2p^{i-1}(p-1), 2p^j] - [2p^{i-1}, 2p^j, 2p^{i-1}(p-1)] + [2p^j, 2p^{i-1}, 2p^{i-1}(p-1)]$. In $A_n^\bullet$, we get

$$\delta(x_i) = \binom{p^i}{p^{i-1}}[2p^i] = \binom{p^i}{p^{i-1}} y_i$$
$$\delta(x_i y_j) = \binom{p^i}{p^{i-1}}([2p^i, 2p^j] - [2p^j, 2p^i]) = \binom{p^i}{p^{i-1}} y_i y_j$$

Hence we can conclude

$$\tilde{\beta}(x_i) = \frac{1}{p}\binom{p^i}{p^{i-1}} y_i \qquad \tilde{\beta}(x_i y_j) = \frac{1}{p}\binom{p^i}{p^{i-1}} y_i y_j.$$

The coefficient

$$\frac{1}{p}\binom{p^i}{p^{i-1}} = \binom{p^i - 1}{p^{i-1} - 1}$$

is an integer congruent to 1 mod p by Lucas' theorem [Fin47]

By a similar, a bit tedious computation we get:

Lemma 2.9. *The differential δ on $A_n^\bullet$ operates as follows:*

$$\delta(x_1^{a_1} \cdots x_k^{a_k} y_0^{b_1} \dots y_l^{b_l}) = \sum_i \binom{p^i}{p^{i-1}} x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_k^{a_k} y_i y_0^{b_0} \cdots y_l^{b_l}$$

Hence the Bocksteins are given by

$$\tilde{\beta}(x_1^{a_1}\cdots x_k^{a_k}y_0^{b_1}\dots y_l^{b_l}) = \frac{1}{p}\sum_i \binom{p^i}{p^{i-1}} x_1^{a_1}\cdots x_i^{a_i-1}\cdots x_k^{a_k}y_i y_0^{b_0}\cdots y_l^{b_l}$$

and

$$\beta(x_1^{a_1}\cdots x_k^{a_k}y_0^{b_1}\dots y_l^{b_l}) = \sum_i x_1^{a_1}\cdots x_i^{a_i-1}\cdots x_k^{a_k}y_i y_0^{b_0}\cdots y_l^{b_l}.$$

As $\beta^2 = 0$, we can look at the *Bockstein cohomology groups*

$$BH^*(A_n^\bullet, \mathbb{Z}_p) = \operatorname{Ker}\beta / \operatorname{Im}\beta.$$

Lemma 2.10. *[Hat02, Cor. 3E.4] The group $H^*(A_n^\bullet, \mathbb{Z})$ contains no element of order p^2 if and only if*

$$\dim_{\mathbb{Z}_p} BH^r(A_n^\bullet, \mathbb{Z}_p) = \operatorname{rk} H^r(A_n^\bullet, \mathbb{Z}).$$

In this case the map

$$H^*(A_n^\bullet, \mathbb{Z}) \to H^*(A_n^\bullet, \mathbb{Z}_p)$$

is injective on the p-torsion and its image is $\operatorname{Im}\beta$*.*

Vainshtein stated that $H^*(A_n^\bullet, \mathbb{Z})$ has no elements of order p^2:

Theorem 2.11. *[Vai78] The integral cohomology is given by*

$$H^0(C_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \qquad\qquad H^1(C_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \text{ if } n \geq 2$$

and

$$H^r(C_n(\mathbb{C}), \mathbb{Z}) = \bigoplus_p \tilde{\beta}_p H^{r-1}(C_n(\mathbb{C}), \mathbb{Z}_p) \text{ for } r \geq 2.$$

Proof. Take any $x \in \operatorname{Ker}\beta$ of the form

$$x = x_j^k f + x_j^{k-1} y_j g$$

for $k \geq 0$, $j > 0$ where f, g do not contain x_j or y_j. We compute

$$\beta(x) = x_j^{k-1} y_j f + x_j^k \beta(f) + x_j^{k-1} y_j \beta(g).$$

Hence we see $\beta(g) = f$ and $\beta(x_j^k g) = x$. So we have shown that

$$\operatorname{Ker}\beta / \operatorname{Im}\beta = \mathbb{Z}_p \otimes \mathbb{Z}_p y_0.$$

□

Remark 2.12. The map β looks suspiciously like a derivation. We will first work with integer coefficients. We consider the free graded commutative $\mathbb{Z}$-algebra

$$\Gamma = \Lambda\langle x_1, x_2, ..., y_0, y_1, ...\rangle \qquad \deg(x_i) = 2p^i - 2 \qquad \deg(y_i) = 2p^i - 1.$$

with the map

$$\beta(x_1^{a_1}\cdots x_k^{a_k}y_0^{b_1}\dots y_l^{b_l}) = \sum_i x_1^{a_1}\cdots x_i^{a_i-1}\cdots x_k^{a_k}y_i y_0^{b_0}\cdots y_l^{b_l}.$$

Take a copy

$$\Gamma' = \Lambda\langle X_1, X_2, ..., Y_0, Y_1, ...\rangle \qquad \deg(X_i) = 2p^i - 2 \qquad \deg(Y_i) = 2p^i - 1.$$

of Γ. We can embedded the abelian group Γ into $\Gamma' \otimes \mathbb{Q}$ via

$$\Gamma \hookrightarrow \Gamma' \otimes \mathbb{Q}, \quad x_1^{a_1}\cdots x_k^{a_k}y_0^{b_1}\dots y_l^{b_l} \mapsto \frac{1}{a_1!}X_1^{a_1}\cdots \frac{1}{a_k!}X_k^{a_k}Y_0^{b_1}\dots Y_l^{b_l}.$$

Write $\star$ for the multiplication on Γ'. Then

$$x_i^{j_1} \star x_i^{j_2} = \binom{j_1 + j_2}{j_1} x_i^{j_1+j_2}$$

and $\star$ induces a multiplication on Γ (a so called divided power algebra [Hat02, Ex 3.5C]). The advantage of $\star$ is that the map $\beta = \beta'_{|\Gamma}$ comes from the unique derivation β' on $\Gamma' \otimes \mathbb{Q}$ defined by

$$\beta'(X_i) = Y_i \qquad\qquad \beta'(Y_i) = 0$$

and the rule (compare [FHT01, Chap. 3])

$$\beta'(z_1 \star z_2) = \beta'(z_1) \star z_2 + (-1)^{\deg z_1} z_1 \star \beta'(z_2).$$

The Bockstein morphism for $A_n^\bullet$ is now the reduction mod p of β.

Corollary 2.13. *We have an isomorphism*

$$p\text{-Torsion of } H^{r+1}(C_\infty(\mathbb{C}), \mathbb{Z}) \simeq \text{ degree } r\text{-part of } \Lambda\langle x_1, x_2, \dots, y_1, y_2, \dots\rangle \otimes \mathbb{Z}_p.$$

for $r > 0$.

Proof. Let $R = \Lambda\langle x_1, x_2, \dots, y_1, y_2, \dots\rangle \otimes \mathbb{Z}_p$. Theorem 2.2 shows that

$$H^*(C_\infty(\mathbb{C}), \mathbb{Z}_p) = R \oplus y_0 R.$$

By lemma 2.9 we know that $\beta(xy_0) = \beta(x)y_0$ and $\beta(R) \subset R$. This shows

$$\operatorname{Im}\beta = \beta(R) \oplus y_0\beta(R).$$

Decompose $R = \beta(R) \oplus R'$. As $\operatorname{Ker}\beta = \operatorname{Im}\beta \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p y_0$, the map

$$\beta(R) \oplus R' \to \beta(R) \oplus y_0\beta(R) = \operatorname{Im}\beta,\ (z_1, z_2) \mapsto \beta(z_2) + y_0 z_1$$

is a bijective map between the degree r part of R and the degree $r+1$ part of $\operatorname{Im}\beta$ for $r > 0$. However, it does not respect the size, so the isomorphism is only possible for $n \to \infty$. $\square$

Remark 2.14. The description of dimension of the p-torsion of $H^r(C_n(\mathbb{C}), \mathbb{Z})$ in [CLM07, Appendix to III] seems to be wrong.

Example 2.15. In table 2, we compute $H^*(C_{24}(\mathbb{C}), \mathbb{Z}_3)_{(3)}$ by applying theorem 2.11 and the formula 2.9 to our example 2.7.

TABLE 2. The 3-torsion in the cohomology group $H^*(C_{24}(\mathbb{C}), \mathbb{Z})$

r	basis of $H^r(C_{24}(\mathbb{C}), \mathbb{Z})_{(3)}$ as $\mathbb{Z}_3$-module
0	–
1	–
2	–
3	–
4	–
5	$y_1 = [6]$
6	$y_0 y_1 = [2, 6] - [6, 2]$
7	–
8	
9	$x_1 y_1 = [2, 4, 6] - [2, 6, 4] + [6, 2, 4]$
10	$x_1 y_0 y_1 = [2, 4, 2, 6] - [2, 4, 6, 2] + [2, 6, 4, 2] - [6, 2, 4, 2]$
11	–
12	–
13	$x_1^2 y_1 = [2, 4, 2, 4, 6] - [2, 4, 2, 6, 4] + [2, 4, 6, 2, 4] - [2, 6, 4, 2, 4] + [6, 2, 4, 2, 4]$
14	$x_1^2 y_0 y_1 = [2, 4, 2, 4, 2, 6] - [2, 4, 2, 4, 6, 2] + [2, 4, 2, 6, 4, 2] - [2, 4, 6, 2, 4, 2,] + \dots$
15	–
16	–
17	$y_2 = [18]$ $x_1^3 y_1 = [2, 4, 2, 4, 2, 4, 6] - \dots$
18	$y_0 y_2 = [2, 18] - [18, 2]$
19	–
20	–
21	$28 x_1 y_2 + x_2 y_1 = 28([2, 4, 18] - [2, 18, 4] + [18, 2, 4]) + [6, 12, 6]$
22	$y_1 y_2 = [6, 18] - [18, 6]$
≥ 23	–

3. Configuration Spaces of the Sphere

We will describe a cellular decompostion of $\overline{C_n(S^2)}$ by Napolitano [Nap03] and show how it can be used to compute the cohomology of $C_n(S^2)$.

3.1. Cellular Decomposition of $\overline{C_n(S^2)}$. Using $S^2 = \mathbb{R}^2 \sqcup \infty$, the cellular decomposition of $\overline{C_n(\mathbb{C})}$ can be extended to a cellular decomposition of $\overline{C_n(S^2)}$ by looking

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & A_n^{r-1} \arrow[r, "\delta"] \arrow[d, "S"] \arrow[dr, "D"] & A_n^{r} \arrow[r, "\delta"] \arrow[d, "S"] \arrow[dr, "D"] & A_n^{r+1} \arrow[r, "\delta"] \arrow[d, "S"] & \cdots \\
\cdots \arrow[r, "\delta"] & A_{n-1}^{r-3} \arrow[r, "\delta"] & A_{n-1}^{r-2} \arrow[r, "\delta"] & A_{n-1}^{r-1} \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

at configurations that do or do not contain ∞. The resulting complex $B_n^\bullet = (B_n^r)$ with $H^*(B_n^\bullet, \mathbb{Z}) = H^*(C_n(S^2), \mathbb{Z})$ has chains

$$B_n^r = A_n^r \oplus A_{n-1}^{r-2} = \mathbb{Z}\,\mathrm{Comb}(n, n-r) \oplus \mathbb{Z}\,\mathrm{Comb}(n-1, n-r+1).$$

The new boundary maps Δ were computed by Napolitano [Nap03]: We define a new operator $D\colon A_n^r \to A_{n-1}^{r-1}$ by

$$\mathrm{D}[n_1, \ldots, n_s] = \sum_{i=1}^{s} Q(n_i)(-1)^{\sum_{j=1}^{i-1} n_i}[n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_s]$$

where

$$Q(n_i) = \begin{cases} 0 & \text{if } n_i \equiv 1 \mod 2 \\ 2 & \text{otherwise.} \end{cases}$$

The differential Δ of the complex $B_n^\bullet$ is then given by

$$\Delta\colon B_n^r \to B_n^{r+1}, (a, b) \mapsto (\delta(a), \delta(b) + (-1)^{n-r}\,\mathrm{D}(a)).$$

Corollary 3.1. *We have $D \equiv 0 \mod 2$ and therefore $B_n^\bullet \otimes \mathbb{Z}_2 = (A_n^\bullet \oplus A_{n-1}^\bullet) \otimes \mathbb{Z}_2$ and*

$$H^r(C_n(S^2), \mathbb{Z}_2) = H^r(C_n(\mathbb{C}), \mathbb{Z}_2) \oplus H^{r-2}(C_{n-1}(\mathbb{C}), \mathbb{Z}_2).$$

3.2. Mapping Cone Complex.

The relation

$$D \circ \delta = \delta \circ D$$

is equivalent to $\Delta^2 = 0$. This means we can see D as a chain map

$$D\colon A_n^\bullet \to A_{n-1}^\bullet[1]$$

and the complex $B_n^\bullet$ can be interpreted as the mapping cone complex of the chain map D. The short exact sequence of chain complexes

$$0 \to A_{n-1}^\bullet[2] \to B_n^\bullet \to A_n^\bullet \to 0.$$

given by $a_2 \mapsto (0, a_2)$ and $(a_1, a_2) \mapsto a_1$ induces a long exact sequence

$$\cdots \to H^{r-1}(A_n^\bullet) \to H^r(A_{n-1}^\bullet[2]) \to H^r(B_n^\bullet) \to H^r(A_n^\bullet) \to H^{r+1}(A_{n-1}^\bullet[2]) \to \ldots.$$

The connecting homomorphism can be identified with D^*.

Lemma 3.2. *We get a long exact sequence*

$$\cdots \to H^{r-1}(A_n^\bullet) \xrightarrow{D^*} H^{r-2}(A_{n-1}^\bullet) \to H^r(B_n^\bullet) \to H^r(A_n^\bullet) \xrightarrow{D^*} H^{r-1}(A_{n-1}^\bullet) \to \ldots$$

We can use this long exact sequence to compare the cohomology of $B_n^\bullet$, $A_n^\bullet$ and $A_{n-1}^\bullet$. Next we will construct a map

$$S\colon A_n^r \to A_{n-1}^{r-2},$$

which is almost a chain homotopy $D \approx 2\delta S + 2S\delta$ between D and the zero map. This allows us to compute the rank of D^*.

4. Construction of (almost) a Null Homotopy

As a motivation we first look at the case $r = n-1$. We set $S[n] = [1, n-2]$. Then we have

$$2\delta S[n] = 2\delta[1, n-2] = 2[n-1] = D[n]$$

if n is even and

$$2\delta S[n] = 2\delta[1, n-2] = 0 = D[n]$$

otherwise.

In general, we define $S\colon A_n^r \to A_{n-1}^{r-2}$ by

$$S[n_1, \dots, n_s] = \sum_{1 \le k \le i \le s} (-1)^{k+1+\sum_{m=1}^{k-1} n_m}[n_1, \dots, n_{k-1}, 1, n_k, \dots, n_{i-1}, n_i - 2, n_{i+1}, \dots, n_s].$$

If $n_i - 2 \le 0$, we simply omit this summand.

Lemma 4.1. *For every composition* $[n_1, \dots, n_s]$ *with* $n_s \neq 2$ *we have*

$$(D - 2\delta \circ S - 2S \circ \delta)[n_1, \dots, n_s] = 0$$

and

$$(D - 2\delta\circ S - 2S\circ\delta)[n_1, \dots, n_{s-1}, 2] = 2\sum_{1 \le k \le s} (-1)^{s+k+\sum_{m=1}^{k-1} n_m}[n_1, \dots, n_{k-1}, 1, n_k, \dots, n_{s-1}]$$

otherwise.

Proof. For convenience we introduce the operators δ_l by

$$\delta_l[m_1, \dots, m_t] = (-1)^{l-1} P(m_l, m_{l+1})[m_1, \dots, m_{l-1}, m_l + m_{l+1}, m_{l+2}, \dots, m_t]$$

and the abbreviations

$$n_{k,i} = (-1)^{k+1+\sum_{m=1}^{k-1} n_m}[n_1, \dots, n_{k-1}, 1, n_k, \dots, n_{i-1}, n_i - 2, n_{i+1}, \dots, n_s].$$

Let us first assume that all $n_i > 2$. We compute

$$\delta \circ S[n_1, \dots n_r] = \sum_{\substack{1 \le l \le s \\ k \le i}} \delta_l(n_{k,i})$$

by splitting up the index set

$$I = \{1 \le l \le s, 1 \le k \le i \le s\}$$

into

$$I = I_1 \sqcup \dots \sqcup I_8$$

where

$$
\begin{aligned}
I_1 &= \{1 \le l < k-1, k \le i\} & I_4 &= \{l = i, k < i\}\\
I_2 &= \{k+1 \le l < i\} & I_5 &= \{l = i+1, k \le i\}\\
I_3 &= \{i+2 \le l \le s, k \le i\} & I_6 &= \{l = k-1, k \le i\}\\
& & I_7 &= \{l = k, k < i\}\\
& & I_8 &= \{l = k = i\}.
\end{aligned}
$$

Now we look at the individual summands $T_j = \sum_{I_j} \delta_l(n_{k,i})$ and expand them after doing some index shifts. Write $\mathrm{ind} = k + l + \sum_{m=1}^{k-1} n_m$.

$$
\begin{aligned}
T_1 &= \sum_{\substack{l<k-1\\ k\le i}} (-1)^{\mathrm{ind}} P(n_l, n_{l+1})[\ldots, n_l + n_{l+1}, \ldots, n_{k-1}, 1, n_k, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots]\\
T_2 &= \sum_{k\le l<i-1} (-1)^{\mathrm{ind}+1} P(n_l, n_{l+1})[\ldots, n_{k-1}, 1, n_k, \ldots, n_l + n_{l+1}, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots]\\
T_3 &= \sum_{k\le i<l} (-1)^{\mathrm{ind}+1} P(n_l, n_{l+1})[\ldots, n_{k-1}, 1, n_k, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots, n_l + n_{l+1}, \ldots]
\end{aligned}
$$

The next terms

$$
\begin{aligned}
T_4 &= \sum_{k<i} (-1)^{k+i+\sum_{m=1}^{k-1} n_m} P(n_{i-1}, n_i - 2)[\ldots, n_{k-1}, 1, n_k, \ldots, n_{i-1} + n_i - 2, n_{i+1}, \ldots]\\
T_5 &= \sum_{k\le i} (-1)^{k+i+1+\sum_{m=1}^{k-1} n_m} P(n_i - 2, n_{i+1})[\ldots, n_{k-1}, 1, n_k, \ldots, n_{i-1}, n_i - 2 + n_{i+1}, \ldots]
\end{aligned}
$$

sum up to

$$
T_4 + T_5 = \sum_{k\le i} (-1)^{k+i+1+\sum_{m=1}^{k-1} n_m} P(n_i, n_{i+1})[\ldots, n_{k-1}, 1, n_k, , \ldots, , n_{i-1}, n_i - 2 + n_{i+1}, \ldots]
$$

where we use the identity $P(x-2, y) + P(x, y-2) = P(x, y)$. Altogether we have

$$
T_1 + T_2 + T_3 + T_4 + T_5 = -S \circ \delta[n_1, \ldots, n_s].
$$

The terms

$$
\begin{aligned}
T_6 &= \sum_{k\le i} (-1)^{2k-2+\sum_{m=1}^{k-1} n_m} P(n_{k-1}, 1)[\ldots, n_{k-2}, n_{k-1} + 1, n_k, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots]\\
T_7 &= \sum_{k<i} (-1)^{2k-1+\sum_{m=1}^{k-1} n_m} P(1, n_k)[\ldots, n_{k-1}, 1 + n_k, n_{k+1}, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots]
\end{aligned}
$$

cancel each other. The remaining summand

$$
T_8 = \sum_i (-1)^{\sum_{m=1}^{i-1} n_m} P(1, n_i - 2)[\ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots]
$$

can be identified with

$$
2T_8 = D[n_1, \ldots, n_s].
$$

Here we use $P(1, n_i - 2) = 1$ if n_i even and $P(1, n_i - 2) = 0$ if n_i odd. In the end we get

$$
2\delta \circ S[n_1, \ldots, n_s] = -2S \circ \delta[n_1, \ldots, n_s] + D[n_1, \ldots, n_s]
$$

In case that $n_j = 2$ with $j < s$, all contributions containing $n_j - 2$ in T_4, T_5 and T_8 are missing in $\delta \circ S$, but not in $S \circ \delta$ and D. So we have to add

$$\begin{aligned} T_4' =& \sum_{k<j} (-1)^{k+j+\sum_{m=1}^{k-1} n_m} P(n_{j-1}, 0)[\dots, 1, n_k, \dots, n_{j-2}, n_{j-1}, n_{j+1}, \dots] \\ T_5' =& \sum_{k\leq j} (-1)^{k+j+1+\sum_{m=1}^{k-1} n_m} P(0, n_{j+1})[\dots, 1, n_k, \dots, n_{j-1}, n_{j+1}, \dots] \\ T_8' =& (-1)^{\sum_{m=1}^{j-1} n_m} P(1, 0)[\dots, n_{j-1}, 1, n_{j+1}, \dots] \end{aligned}$$

which simplifies using $P(x, 0) = 1$ to:

$$\begin{aligned} T_4' + T_8' =& \sum_{k\leq j} (-1)^{k+j+\sum_{m=1}^{k-1} n_m} [\dots, n_{k-1}, 1, n_k, n_{j-2}, \dots, n_{j-1}, n_{j+1}, \dots] \\ T_5' =& \sum_{k\leq j} (-1)^{k+j+1+\sum_{m=1}^{k-1} n_m} [\dots, n_{k-1}, 1, n_k, \dots, n_{j-1}, n_{j+1}, , \dots] \end{aligned}$$

Hence we have

$$(D - 2\delta \circ S - 2S \circ \delta)[n_1, \dots, n_s] = 2T_4' + 2T_5' + 2T_6' = 0,$$

if $n_j = 2$ with $j < s$. In the case $n_s = 2$, we get

$$\begin{aligned} &(D - 2\delta \circ S - 2S \circ \delta)[n_1, \dots, n_{s-1}, 2] \\ &= 2T_4' + 2T_8' \\ &= 2 \sum_{1\leq k\leq s} (-1)^{s+k+\sum_{m=1}^{k-1} n_m} [n_1, \dots, n_{k-1}, 1, n_k, \dots, n_{s-1}]. \end{aligned}$$

A similar argument deals with the case that some $n_j = 1$. □

Lemma 4.2. *For every partition $[n_1, \dots, n_s]$ with all n_i even we have*

$$(D - 2\delta \circ S - 2S \circ \delta) \operatorname{Ins}_t[n_1, \dots, n_{s-1}, 2] = 2(t+1)(-1)^{t+1} \operatorname{Ins}_{k+1}[n_1, \dots, n_{s-1}].$$

PROOF. Take any $I \subset \{1, \dots, s+t\}$ with $|I| = t+1$. The coefficient of the term $\operatorname{Ins}_I[n_1, \dots, n_{s-1}]$ in $(D - 2\delta \circ S - 2S \circ \delta) \operatorname{Ins}_t[n_1, \dots, n_{s-1}, 2]$ is given by

$$2(-1)^{st+t} \sum_{i\in I} (-1)^{i+\sum_{j\in I, j<i} 1+\sum_{j\in I, j<i} j+\sum_{j\in I, j>i}(j-1)} = 2(-1)^{s(t+1)}(t+1)(-1)^{\sum_{j\in I} j}.$$

This is the coefficient of $\operatorname{Ins}_I[n_1, \dots, n_{s-1}]$ in $2(t+1)(-1)^{t+1} \operatorname{Ins}_{t+1}[n_1, \dots, n_{s-1}]$. □

Corollary 4.3. *Let $p > 2$. Take $x_1^{c_1} \cdots x_k^{c_k} y_1^{d_1} \cdots y_l^{d_l} y_0$ with size m. Then*

$$(D - 2\delta \circ S - 2S \circ \delta)(x_1^{c_1} \cdots x_k^{c_k} y_1^{d_1} \cdots y_l^{d_l}) = 0$$

and

$$(D - 2\delta \circ S - 2S \circ \delta)(x_1^{c_1} \cdots x_k^{c_k} y_1^{d_1} \cdots y_l^{d_l} y_0) = 2(-1)^{n-m+1}(n-m+1)x_1^{c_1} \cdots x_k^{c_k} y_1^{d_1} \cdots y_l^{d_l}.$$

Corollary 4.4. *Let $p = 2$. Take $x_1^{c_1} \cdots x_k^{c_k} y_1^{d_1} \cdots y_l^{d_l} y_0$ with size m. Then*

$$(D - 2\delta \circ S - 2S \circ \delta)(x_2^{c_2} \cdots x_k^{c_k} y_1^{d_1} \cdots y_l^{d_l}) = 0$$

and if $c_1 > 0$

$(D-2\delta\circ S-2S\circ\delta)(x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}) = 2(-1)^{n-m+3}(n-m+3)x_1^{c_1-1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}y_0.$

Furthermore,

$(D-2\delta\circ S-2S\circ\delta)(x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}y_0) = 2(-1)^{n-m+1}(n-m+1)x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}.$

This allows us to compute the map $D^*\colon H^i(A_n^\bullet) \to H^{i-1}(A_{n-1}^\bullet)$ with both $\mathbb{Z}$ and $\mathbb{Z}_p$-coefficients.

5. Proof of Main Theorem

Proof of Th. 1.1. By lemma 4.1 and corollary 4.3 we can conclude that the rank of the map $D^*\colon H^r(A_n^\bullet, \mathbb{Z}_p) \to H^{r-1}(A_{n-1}^\bullet, \mathbb{Z}_p)$ is given by the number of monomials

$$x_1^{c_1}\dots x_k^{c_k}y_0y_1^{d_1}\dots y_l^{d_l}$$

of degree r and size $m \le n$ such that $p \nmid 2(n-m+1)$. Equivalently, the rank can be written as

$$B_p'(n,r) = \left|\left\{\begin{matrix}1 \le a_1 \le a_2 \le \cdots \le a_g \\ 1 \le b_1 < b_2 < \cdots < b_h\end{matrix}\;\middle|\;\begin{matrix}2\sum_i p^{a_i} + 2\sum_j p^{b_j} + 1 - 2g - h = r \\ 2\sum_i p^{a_i} + 2\sum_j p^{b_j} + 2 \le n \\ p \nmid 2(n - 2\sum_i p^{a_i} - 2\sum_j p^{b_j} - 1)\end{matrix}\right\}\right|$$

By the long exact sequence of lemma 3.2 we have determined

$$\dim H^r(C_n(S^2), \mathbb{Z}_p) = B_p(n,r) + B_p(n-1,r-2) - B_p'(n,r) - B_p'(n,r-1).$$

$\square$

Corollary 5.1. *[Sal04] This can be written as a generating series. Let*

$$Q = \prod_{i>0} \frac{1 + w^{2p^i-1}z^{2p^i}}{1 - w^{2p^i-2}z^{2p^i}}.$$

Then we have for $p > 2$*:*

$$\sum_{r,n\ge 0} \dim H^r(C_n(S^2), \mathbb{Z}_p)\, w^r z^n = \left(\frac{1}{1-z} + \frac{wz^{p+1}}{1-z^p} + \frac{w^3z^3}{1-z} + \frac{w^2z}{1-z^p}\right) Q$$

Corollary 5.2. *Our description implies eventual periodicity*

$$\dim H^r(C_{n+p}(S^2), \mathbb{Z}_p) = \dim H^r(C_n(S^2), \mathbb{Z}_p)$$

if $n \ge 2r$*.*

Proof. As $\sum_{i=1}^g p^{a_i} + \sum_{j=1}^h p^{b_j} \ge 2g + h$, we get the inequalities $r \ge 2g + h + 1$ and $\sum_{i=1}^g p^{a_i} + \sum_{j=1}^h p^{b_j} \le 2r - 2$. Hence we have for $n \ge 2r + 2$:

$$B_p(n,r) = B_p(n+1,r) \qquad\qquad B_p'(n+p,r) = B_p'(n,r)$$

$\square$

Proof of Th. 1.3. For $n \leq 3$, we can easily check the theorem by hand. Take $n \geq 4$. We look at the beginning of the long exact sequence of lemma 3.2. We immediately read off

$$H^0(B_n^\bullet) \simeq H^0(A_n^\bullet).$$

As $H^2(A_n^\bullet) = H^2(A_{n-1}^\bullet) = 0$ by application of lemma 2.11, we get the exact sequence

$$0 \to H^1(B_n^\bullet) \to H^1(A_n^\bullet) \xrightarrow{D^*} H^0(A_{n-1}) \to H^2(B_n^\bullet) \to 0.$$

The group $H^1(A_n^\bullet) = \mathbb{Z}$ is generated by the class of y_0 and the group $H^0(A_{n-1}^\bullet) = \mathbb{Z}$ is generated by the class 1 with the map $D^*(y_0) = (2n-2) \cdot 1$ by lemma 4.3. Hence we see

$$H^1(B_n^\bullet) = 0 \qquad\qquad H^2(B_n^\bullet) = \mathbb{Z}/(2n-2)\mathbb{Z}.$$

If we had $D = 2\delta \circ S + 2S \circ \delta$, we would have a chain map

$$A_n^\bullet \to B_n^\bullet,\ a \mapsto (a, -2(-1)^{n-r} S(a)),$$

that would split the sequence

$$0 \to A_{n-1}^\bullet[2] \to B_n^\bullet \to A_n^\bullet \to 0,\ a_2 \mapsto (0, a_2),\ (a_1, a_2) \mapsto a_1$$

on the right.

In our case, the long exact sequence of lemma 3.2 gives us short exact sequences

$$0 \to \operatorname{Coker} D^* \to H^r(B_n^\bullet) \to \operatorname{Ker} D^* \to 0.$$

We want to construct a right splitting $s : \operatorname{Ker} D^* \to H^r(B_n^\bullet)$. For $r \geq 2$, the cohomology group $H^r(A_n^\bullet)$ is finite and has no elements of order p^2. For every prime p, we can take a $\mathbb{Z}_p$-basis of the p-torsion in $\operatorname{Ker} D^*$ consisting of the classes $\overline{b_i}$ of the chains

$$b_i = \tilde{\beta}(m_i) = \frac{1}{p}\delta(m_i)$$

for some monomials $m_i = x_1^{a_1} \dots x_k^{a_k} y_1^{b_1} \dots y_l^{b_l} y_0^{b_0} \in A_n^\bullet$. By corollary 4.3, we can find integers k_i and monomials m_i' such that

$$(D - 2S \circ \delta - 2S\delta \circ S)(m_i) = k_i p m_i'.$$

If $p \neq 2$ and $y_0 \mid m_i$, we have $m_i' = x_1^{a_1} \dots x_k^{a_k} y_1^{b_1} \dots y_l^{b_l}$. Define $E = D - 2S \circ \delta - 2\delta \circ S$. Observe that $E \circ S = S \circ E$. Hence we get

$$E(m_i) = pk_i m_i' \qquad\qquad E(b_i) = k_i \delta(m_i').$$

Define a map

$$s\colon \operatorname{Ker} D^* \to H^r(B_n^\bullet, \mathbb{Z})$$

by setting

$$s(\bar{b_i}) = \left(b_i,\ -2(-1)^{n-r} S(b_i) - (-1)^{n-r} k_i m_i'\right).$$

We see that

$$\begin{aligned} \Delta \circ s(\bar{b_i}) &= \left(\delta(b_i),\ -2(-1)^{n-r}\delta \circ S(b_i) + (-1)^{n-r} D(b_i) - (-1)^{n-r} k_i \delta(m_i')\right) \\ &= \left(\delta(b_i),\ 2(-1)^{n-r} S \circ \delta(b_i) + (-1)^{n-r} E(b_j) - (-1)^{n-r} k_i \delta(m_i')\right) \\ &= 0 \end{aligned}$$

and hence $s(\bar{b_i})$ is a cycle in $H^r(B_n^\bullet, \mathbb{Z})$. We have to show that $ps(\bar{b_i})$ is a boundary. We have $pb_i = \delta(m_i)$ and can compute

$$\begin{aligned} ps(\bar{b_i}) &= \left(pb_i,\, -2(-1)^{n-r} S(pb_i) - (-1)^{n-r} pk_i m_i'\right) \\ &= \left(\delta(m_i),\, -2(-1)^{n-r} S \circ \delta(m_i) - (-1)^{n-r} pk_i m_i'\right) \\ &= \left(\delta(m_i),\, (-1)^{n-r}(2\delta \circ S(m_i) - D(m_i) + E(m_i) - pk_i m_i')\right) \\ &= \left(\delta(m_i),\, 2(-1)^{n-r}\delta \circ S(m_i) - (-1)^{n-r} D(m_i)\right) \\ &= \Delta\left(m_i,\, S(m_i)\right). \end{aligned}$$

Hence s is a well-defined right splitting of the sequence

$$0 \to \operatorname{Coker} D^* \to H^r(B_n^\bullet) \to \operatorname{Ker} D^* \to 0.$$

For $r \geq 3$, both $\operatorname{Ker} D^*$ and $\operatorname{Coker} D^*$ have no elements of p^2, thus the same is true for $H^r(B_n^\bullet)$. □

Example 5.3. We want to compute the 3-torsion in the groups $H^6(C_9(S^2), \mathbb{Z})$ and $H^6(C_{10}(S^2), \mathbb{Z})$. We use the long exact sequence

$$\cdots \to H^5(A_n^\bullet) \xrightarrow{D^*} H^4(A_{n-1}^\bullet) \to H^6(B_n^\bullet) \to H^6(A_9^\bullet) \xrightarrow{D^*} H^5(A_{n-1}^\bullet) \to \dots$$

For $p = 3$, the generators of $H^*(A_n^\bullet, \mathbb{Z}_3)$ are:

generator	x_1	x_2	y_0	y_1	y_2	$\dots$
degree	4	16	1	5	17	$\dots$
size	6	18	2	6	18	$\dots$

So

$$H^6(A_9^\bullet, \mathbb{Z}_3) = H^6(A_{10}^\bullet, \mathbb{Z}_3) = \mathbb{Z}_3 y_0 y_1.$$

and

$$H^4(A_9^\bullet, \mathbb{Z}_3) = H^4(A_{10}^\bullet, \mathbb{Z}_3) = \mathbb{Z}_3 x_1$$

We have $D^*(y_0 y_1) = 2(n-7)y_1$ and $D^*(x_1 y_0) = 2(n-7)x_1$. Hence we get

$$H^6(B_9^\bullet, \mathbb{Z}_3) = 0 \qquad H^6(B_{10}^\bullet, \mathbb{Z}_3) = \mathbb{Z}_3^2.$$

The Bockstein $\tilde{\beta}(x_1 y_0) = y_0 y_1$ shows

$$H^6(A_9^\bullet, \mathbb{Z})_{(3)} = H^6(A_{10}^\bullet, \mathbb{Z})_{(3)} = \mathbb{Z}_3 y_0 y_1$$

and

$$H^4(A_9^\bullet, \mathbb{Z})_{(3)} = H^4(A_{10}^\bullet, \mathbb{Z})_{(3)} = 0.$$

We get

$$H^6(B_9^\bullet, \mathbb{Z})_{(3)} = 0 \qquad H^6(B_{10}^\bullet, \mathbb{Z})_{(3)} = \mathbb{Z}_3.$$

6. Some Tables

The tables 3 and 4 were computed with the help of the computer algebra systems Sage [Sage] and Magma [BCP97]. The cohomology groups $H^r(C_n(S^2), \mathbb{Z})$ have already been determined for $n \leq 9$ by Sevryuk [Sev84] and Napolitano [Nap03].

TABLE 3. Cohomology groups $H^i(C_n(\mathbb{C}), \mathbb{Z})$

$n \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$\mathbb{Z}$															
2, 3	$\mathbb{Z}$	$\mathbb{Z}$														
4, 5	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$												
6, 7	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$										
8, 9	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_3$	$\mathbb{Z}_2$								
10, 11	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_5$						
12, 13	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2\times\mathbb{Z}_5$					
14, 15	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2$	0	$\mathbb{Z}_7$		
16, 17	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_2$

TABLE 4. Cohomology groups $H^i(C_n(S^2), \mathbb{Z})$

$n \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$\mathbb{Z}$	0	$\mathbb{Z}$													
2	$\mathbb{Z}$	0	$\mathbb{Z}_2$													
3	$\mathbb{Z}$	0	$\mathbb{Z}_4$	$\mathbb{Z}$												
4	$\mathbb{Z}$	0	$\mathbb{Z}_6$	$\mathbb{Z}\times\mathbb{Z}_2$												
5	$\mathbb{Z}$	0	$\mathbb{Z}_8$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}_2$										
6	$\mathbb{Z}$	0	$\mathbb{Z}_{10}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_3$										
7	$\mathbb{Z}$	0	$\mathbb{Z}_{12}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_3$								
8	$\mathbb{Z}$	0	$\mathbb{Z}_{14}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$								
9	$\mathbb{Z}$	0	$\mathbb{Z}_{16}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$						
10	$\mathbb{Z}$	0	$\mathbb{Z}_{18}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_5$						
11	$\mathbb{Z}$	0	$\mathbb{Z}_{20}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_5$	$\mathbb{Z}_2$	$\mathbb{Z}_5$				
12	$\mathbb{Z}$	0	$\mathbb{Z}_{22}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2$	0				
13	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_5$			
14	$\mathbb{Z}$	0	$\mathbb{Z}_{26}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}_5$	$\mathbb{Z}_7$		
15	$\mathbb{Z}$	0	$\mathbb{Z}_{28}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2\times\mathbb{Z}_7$	0	$\mathbb{Z}_7$
16	$\mathbb{Z}$	0	$\mathbb{Z}_{30}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^4\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^4\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2\times\mathbb{Z}_7$	0	$\mathbb{Z}_2$

CHAPTER 4

Configurations of Points with Sum 0

For any complex quasi-projective algebraic variety X, the *virtual Poincaré polynomial* $S(X) \in \mathbb{Z}[x]$ is defined [Tot02] by the properties

- $S(X) = \sum \operatorname{rk} H^i(X)\, x^i$ for smooth, projective X,
- $S(X) = S(X \setminus C) + S(C)$ for a closed subvariety $C \subset X$,
- $S(X \times Y) = S(X)S(Y)$.

Let E be an elliptic curve with neutral element 0. We will compute the virtual Poincaré polynomial of the space

$$F_n^0(E) = \{x_1, \ldots, x_n \,|\, x_i \neq x_j \text{ and } \sum x_i = 0\}.$$

Our approach is to decompose $F_n(X)$ in the Grothendieck ring of varieties. We use an elementary version of methods of Getzler that immediately generalizes to $F_n^0(E)$. The answer seems to be new.

The combinatorial tools are Stirling numbers and Möbius functions and we will review them first.

1. Stirling Numbers of First Kind

The *Stirling number of first kind* $s(n,k)$ counts the numbers of permutations in S_n with exactly k cycles (compare [Sta11, Chap. 1.3]). Write $\operatorname{Part}(n,k)$ for all the partitions σ of the set $\{1, \ldots, n\}$ into k disjoint, non-empty subsets σ_i. We call the k subsets σ_1, ..., σ_k in no particular order. Then

$$s(n,k) = \sum_{\sigma \in \operatorname{Part}(n,k)} \prod (|\sigma_i| - 1)!.$$

Let x be a positive integer. In order to determine a generating series for $s(n,k)$, we look at the action of S_n on sets of functions $\{1, \ldots, n\} \to \{1, \ldots, x\}$. The quotient consists of the multisets of size n on $\{1, \ldots, x\}$ and has cardinality

$$\binom{n+x-1}{n} = \frac{x(x+1)\cdots(x+n-1)}{n!}.$$

On the other hand, any $\tau \in S_n$ with k cycles has x^k fixed points. By Burnside's lemma

$$\frac{x(x+1)\cdots(x+n-1)}{n!} = \frac{1}{n!} \sum_{\tau \in S_n} |\operatorname{Fix} \tau|$$

and we get

$$x(x+1)\cdots(x+n-1) = \sum s(n,k) x^k.$$

As it is true for all integers x, we have found a formal generating series.

2. The Möbius Function of the Partition Poset

We write $\mathrm{Part}(n)$ for the partitions of the set $\{1, \ldots, n\}$. The number of parts of $\sigma \in \mathrm{Part}(n)$ is called $l(\sigma)$. The set $\mathrm{Part}(n)$ is partially ordered by setting $\sigma \leq \pi$ if σ is finer than π. Write $\mathbb{0} = \{\{1\}, \ldots, \{n\}\}$ for the minimal partition.

Theorem 2.1 (Möbius Inversion)**.** *For any finite poset $(M, \leq)$, the Möbius function $\mu\colon M \times M \to \mathbb{Z}$ on M is defined by the relations*

$$\mu(x, z) = 0 \text{ when } x \not\leq z \qquad \sum_{x \leq y \leq z} \mu(x, y) = \delta(x, z) \text{ when } x \leq z.$$

Here δ is the Kronecker delta

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Let $f\colon M \to \mathbb{Z}$ a function on M and

$$g(x) = \sum_{x \leq y} f(y).$$

Then we can reconstruct f from g:

$$f(x) = \sum_{x \leq y} \mu(x, y) g(y).$$

Following [BG75], we will use Möbius inversion to compute the Möbius function for the poset of partitions. Let x be a positive integer and $p\colon \{1, \ldots, n\} \to \{1, \ldots, x\}$ a function. The preimages of the elements of $\{1, \ldots, x\}$ induce a partition of $\{1, \ldots, n\}$, that we call the kernel of p. Let $f(\sigma)$ be the number of functions $\{1, \ldots, n\} \to \{1, \ldots, x\}$ with kernel σ. Then $f(\mathbb{0})$ counts all injective functions $\{1, \ldots, n\} \to \{1, \ldots, x\}$, hence it is

$$f(\mathbb{0}) = x(x-1)\cdots(x-n+1).$$

On the other hand $g(\sigma) = \sum_{\sigma \leq \pi} f(\pi)$ allows the same values on different parts on σ. Hence we have

$$g(\sigma) = x^{l(\sigma)}.$$

By Möbius inversion

$$f(\mathbb{0}) = \sum_{\sigma} \mu(\mathbb{0}, \sigma) g(\sigma)$$

or

$$x(x-1)\ldots(x-n+1) = \sum_{\sigma \in \mathrm{Part}(n)} \mu(\mathbb{0}, \sigma) x^{l(\sigma)}.$$

As this holds for all values of x, it is valid as an identity for formal polynomials. So for the maximal partion $\mathbb{1} = \{1, \ldots, n\}$, we can immediately read off the constant term and get

$$\mu(\mathbb{0}, \mathbb{1}) = (-1)^{n-1}(n-1)!.$$

For general σ, the poset $\{\pi \in \mathrm{Part}(n) | \pi \leq \sigma\}$ is a product of posets

$$\{\pi \in \mathrm{Part}(n) | \pi \leq \sigma\} \simeq \{\pi \in \mathrm{Part}(|\sigma_1|) | \pi \leq \sigma_1\} \times \cdots \times \{\pi \in \mathrm{Part}(|\sigma_{l(\sigma)}|) | \pi \leq \sigma_{l(\sigma)}\}$$

and hence

$$\mu(\mathbb{0},\sigma)=\mu(\mathbb{0},\sigma_1)\cdots\mu(\mathbb{0},\sigma_{l(\sigma)})=(-1)^{n-l(\sigma)}\prod_i(|\sigma_i|-1)!.$$

3. Virtual Poincaré Polynomials of Configuration Spaces

For any X, we write $[X]$ for the class of X in the Grothendieck ring of varieties. We have maps

$$F_n(X)\to F_{n-1}(X)$$

with fiber $X\setminus(n-1)$ [FH01]. This suggests – ignoring possible topological problems –

$$[F_n(X)]=[F_{n-1}(X)]\times[X-(n-1)]$$

and hence

$$[F_n(X)=[X]([X]-1)\cdots([X]-n+1)=\sum_{k\geq 0}[X]^k(-1)^{n-k}s(n,k).$$

We will prove this formula be a different approach using the Möbius function of the partition poset. It is insprired by Getzler [Get95] [Get99], who even gave a description for the S_n action on $S(F_n(X))$.

We look at the higher diagonals

$$\Delta_\sigma=\{x_1,\dots,x_n\in X^n|\,x_i=x_j\text{ if }i\text{ and }j\text{ are in the same part of }\sigma\}$$

for any partition σ of $\{1,\dots,n\}$. By the inclusion-exclusion principle we have a decomposition

$$[F_n(X)]=[X^n]-\sum_{i\neq j}[\{x_i=x_j\}]+\cdots=\sum_{\sigma\in\mathrm{Part}(n)}m_\sigma[\Delta_\sigma].$$

for some coefficients $m_\sigma\in\mathbb{Z}$. In order to be a valid decomposition of $F_n(x)$, the coefficients m_σ have to satisfy the condition

$$\sum_{\Delta_\pi\subseteq\Delta_\sigma}m_\sigma=\begin{cases}1 & \text{if }\pi=0\\ 0 & \text{otherwise}\end{cases}$$

for any partition $\pi\in\mathrm{Part}(n)$. As $\Delta_\pi\subseteq\Delta_\sigma$ if and only if $\sigma\leq\pi$, these equations are exactly the definition of the Möbius function for the poset $\mathrm{Part}(n)$:

$$\sum_{\sigma\leq\pi}\mu(\mathbb{0},\sigma)=\begin{cases}1 & \pi=\mathbb{0},\\ 0 & \text{otherwise.}\end{cases}$$

So we get

$$m_\sigma=\mu(\mathbb{0},\sigma)=(-1)^{n-l(\sigma)}\prod_i(|\sigma_i|-1)!$$

and with $[\Delta_\sigma]=[X]^{l(\sigma)}$ we can compute:

$$[F_n(X)]=\sum_{\sigma\in\mathrm{Part}(n)}[X]^{l(\sigma)}(-1)^{n-l(\sigma)}\prod_i(|\sigma_i|-1)!=\sum_{k\geq 1}[X]^k(-1)^{n-k}s(n,k).$$

Now applying S immediately proves:

$$S(F_n(X))=\sum_{k\geq 1}S(X)^k(-1)^{n-k}s(n,k)$$

4. Configurations of Points with Sum 0

Let E be an elliptic curve with neutral element 0. There is a map

$$\Sigma\colon F_n(X) \to E,\, (x_1, \ldots, x_n) \mapsto \sum x_i.$$

We look at the fiber $F_n^0(E) = \Sigma^{-1}(0) = \{x_1, \ldots, x_n \in E^n | x_i \neq x_j,\, \sum x_i = 0.\}$

By intersecting the decomposition $[F_n(E)] = \sum_\sigma m_\sigma [\Delta_\sigma]$ with $\Sigma^{-1}(0)$ we get

$$[F_n^0(E)] = \sum_{\sigma \in \mathrm{Part}(n)} m_\sigma \, [\Delta_\sigma \cap \Sigma^{-1}(0)].$$

These loci have a simpler description. Take a partion σ with l parts. We see:

$$\Delta_\sigma \cap \Sigma^{-1}(0) = \{y_1, \ldots, y_l \in E^l | \sum |\sigma_i| \, y_i = 0\}$$

By a coordinate change, we can compute the following solutions of this linear equation:

$$\begin{aligned}\{y_1, \ldots, y_l \in E^l | \sum |\sigma_i| y_i = 0\} &\simeq \{z_1, \ldots, z_l \in E^l | \gcd(|\sigma_1|, \ldots, |\sigma_l|) z_l = 0\} \\ &\simeq E^{l-1} \times (\mathbb{Z}/\gcd(|\sigma_1|, \ldots, |\sigma_l|)\mathbb{Z})^2\end{aligned}$$

With the notation

$$\gcd(\sigma) = \gcd(|\sigma_1|, \ldots, |\sigma_{r(\sigma)}|)$$

we get

$$[F_n^0(E)] = \sum_{\sigma \in \mathrm{Part}(n)} (-1)^{n-l(\sigma)} [E]^{l(\sigma)-1} \gcd{}^2(\sigma) \prod_i (|\sigma_i| - 1)!$$

Hence the following theorem is proven.

Theorem 4.1. *Define*

$$s_m(n,k) = \sum_{\sigma \in \mathrm{Part}(n,k)} \gcd{}^2(\sigma) \prod_i (|\sigma_i| - 1)!.$$

Then we have

$$[F_n^0(E)] = \sum_{k \geq 1} [E]^{k-1} (-1)^{n-k} s_m(n,k)$$

and

$$S(F_n^0(E)) = \sum_{k \geq 1} S(E)^{k-1} (-1)^{n-k} s_m(n,k).$$

The numbers $s_m(n,k)$ are a form of modified Stirling numbers. Any $\sigma \in \mathrm{Part}(n)$ with $l(\sigma) > \frac{n}{2}$ contains a part of length 1. So $\gcd(\sigma) = 1$ and

$$s(n,k) = s_m(n,k) \text{ if } k > \frac{n}{2}.$$

For a prime p, the only partition $\sigma \in \mathrm{Part}(p)$ with $\gcd(p) \neq 1$ is $\sigma = \{\{1, \ldots, p\}\}$. Hence

$$s(p,k) = s_m(p,k) \text{ for } k > 1.$$

In general,

$$s(n,1) = (n-1)! \qquad\qquad s_m(n,1) = n^2 (n-1)!,$$

as $\{\{1, \ldots, n\}\}$ is the only partition of length 1.

Unfortunately, it is not straightforward to extend the methods of [Get95], [Get99] to describe the S_n-action on $S(F_n^0(E))$, because the identification

$$\{y_1, \ldots, y_l \in E^l | \sum |\sigma_i| y_i = 0\} \simeq E^{l-1} \times (\mathbb{Z}/\gcd(\sigma)\mathbb{Z})^2$$

is not compatible with the S_n and S_l actions.

5. Tables

Here we give the full formulas for $[F_n(E)]$ and $[F_n^0(E)]$ for all $n \leq 8$.

n	$[F_n(E)]$
2	$E^2 - E$
3	$E^3 - 3E^2 + 2E$
4	$E^4 - 6E^3 + 11E^2 - 6E$
5	$E^5 - 10E^4 + 35E^3 - 50E^2 + 24E$
6	$E^6 - 15E^5 + 85E^4 - 225E^3 + 274E^2 - 120E$
7	$E^7 - 21E^6 + 175E^5 - 735E^4 + 1624E^3 - 1764E^2 + 720E$
8	$E^8 - 28E^7 + 322E^6 - 1960E^5 + 6769E^4 - 13132E^3 + 13068E^2 - 5040E$

n	$[F_n^0(E)]$
2	$E - 4$
3	$E^2 - 3E + 18$
4	$E^3 - 6E^2 + 20E - 96$
5	$E^4 - 10E^3 + 35E^2 - 50E + 600$
6	$E^5 - 15E^4 + 85E^3 - 270E^2 + 864E - 4320$
7	$E^6 - 21E^5 + 175E^4 - 735E^3 + 1624E^2 - 1764E + 35280$
8	$E^7 - 28E^6 + 322E^5 - 1960E^4 + 7084E^3 - 16912E^2 + 42048E - 322560$

CHAPTER 5

Configuration Spaces of $\mathbb{C} \setminus k$

We look at the cohomology of ordered and unordered configuration spaces of $\mathbb{C} \setminus k$. We compute their normal and virtual Poincaré Polynomials by existing methods and see that Stirling and pyramidal numbers show up. The calculation for $C_n(\mathbb{C} \setminus k)$ seems not to be in the literature in this form. We write P for the ordinary and S for the virtual Poincaré polynomials.

1. Pyramidal Numbers

The *k-dimensional pyramidal numbers* are integers $P_{k,i}$ for $i \geq -1$, $k \geq -1$. They satisify the recursions

$$P_{-1,i} = \begin{cases} 1 & i = 0 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad P_{k+1,i} = \sum_{j=0}^{i} P_{k,j}.$$

An equivalent recursion would be

$$P_{k,0} = 1 \qquad\qquad P_{k+1,i+1} = P_{k,i+1} + P_{k+1,i}.$$

Some examples are

$$P_{0,i} = 1 \qquad\qquad P_{1,i} = i + 1 \qquad\qquad P_{2,i} = \frac{(i+1)(i+2)}{2}.$$

The recursion allows us to compute the generating function

$$\sum P_{k,i} x^i = (1 + x + x^2 + x^3 + x^4 + \dots)^{k+1} = \frac{1}{(1-x)^{k+1}}.$$

Some pyramidal numbers $P_{k,i}$:

k \ i	0	1	2	3	4
-1	1	0	0	0	0
0	1	1	1	1	1
1	1	2	3	4	5
2	1	3	6	10	15
3	1	4	10	20	35

By standard manipulation of generating series for $k \geq 0$:

$$\frac{1}{(1-x)^{k+1}} = \frac{1}{k!}\frac{d^k}{dx^k}\frac{1}{1-x} = \frac{1}{k!}\sum_{i \geq 0}(i+k)\dots(i+2)(i+1)x^i = \sum_{i \geq 0}\binom{i+k}{i}x^i$$

The result

$$P_{k,i} = \binom{i+k}{i}$$

also holds for $k = 0$ and can be proved directly using the recursion

$$P_{k+1,i+1} = \binom{i+k+2}{i+1} = \binom{i+k+1}{i+1} + \binom{i+k+1}{i} = P_{k,i+1} + P_{k+1,i}.$$

The definition could be extended by setting

$$P_{k,i} = 0 \text{ for } i < 0.$$

In this way, all recursions stay valid for $i < 0$.

2. Poincaré Polynomials of $C_n(\mathbb{C} \setminus k)$

Let M be a connected manifold. Napolitano [Nap03, Theorem 2] proved the following relation between the cohomology of unordered configuration spaces of $M \setminus 1$ and $M \setminus 2$:

$$H^j(C_n(M \setminus 2), \mathbb{Z}) = \bigoplus_{t=0}^{n} H^{j-t}(C_{n-t}(M \setminus 1, \mathbb{Z})).$$

We use the conventions

$$H^0(C_0(M \setminus 1), \mathbb{Z}) = \mathbb{Z} \qquad\qquad H^j(C_0(M \setminus 1), \mathbb{Z}) = 0 \text{ if } j > 0.$$

In general, this relation does not hold between the cohomology of the configuration spaces of $M \setminus 1$ and M as the proof works by pushing in points from the missing point.

Theorem 2.1. *We have*

$$\operatorname{rk} H^i(C_n(\mathbb{C} \setminus k), \mathbb{Z}) = \begin{cases} P_{k-1,i} & i = n \\ P_{k-1,i} + P_{k-1,i-1} & 0 \leq i < n \\ 0 & \textit{otherwise} \end{cases}$$

or

$$\sum_{n \geq 0} P(C_n(\mathbb{C} \setminus k))y^n = \frac{1 + xy^2}{(1-y)(1-xy)^k}.$$

Proof. Write

$$Q_k(x,y) = \sum_{n,i \geq 0} \operatorname{rk} H^i(C_n(\mathbb{C} - k), \mathbb{Z})\, x^i y^n.$$

Then applying Napolitano's recursion to $M = S^2 \setminus k+1$ we get

$$Q_{k+1}(x,y) = \frac{Q_k(x,y)}{1-xy}.$$

Arnold's computation of the cohomology of $C_n(\mathbb{C})$ in theorem 3.1 [Arn70] provides initial values for $k = 0$:

$$Q_0(x,y) = 1 + y + (1+x)y^2 + (1+x)y^3 + \cdots = \frac{1+xy^2}{1-y}$$

Hence we have shown

$$Q_k(x,y) = \frac{1+xy^2}{(1-y)(1-xy)^k}.$$

Expansion now proves the theorem. □

This theorem can also be deduced from [DK16, Prop. 3.5]. As $C_1(\mathbb{C} \setminus k) = \mathbb{C} \setminus k$, the reality check for $n = 1$ works:

$$\operatorname{rk} H^j(C_1(\mathbb{C} \setminus k), \mathbb{Z}) = \begin{cases} 1 & \text{for } j = 0 \\ k & \text{for } j = 1 \\ 0 & \text{otherwise} \end{cases}.$$

We can conclude that $\operatorname{rk} H^j(C_n(\mathbb{C} \setminus k), \mathbb{Z})$ stabilizes (seen as a function of n) for $n > j$.

Corollary 2.2. *In the limit we get*

$$\operatorname{rk} H^j(C_\infty(\mathbb{C} \setminus k), \mathbb{Z}) = P_{k-1,j} + P_{k-1,j-1}$$

or as a generating series

$$P(C_\infty(\mathbb{C} \setminus k)) = \frac{1+x}{(1-x)^k}.$$

Taking stability for granted, this can be deduced by the stable version of Napolitano's recursion:

$$H^j(C_\infty(\mathbb{C} \setminus k+1), \mathbb{Z}) = \bigoplus_{t=0}^{j} \operatorname{rk} H^t(C_\infty(\mathbb{C} \setminus k), \mathbb{Z}).$$

Vershinin [Ver99, Cor. 11.1] showed that

$$H^*(C_\infty(\mathbb{C} \setminus k) \simeq H^*(\Omega^2 S^3) \otimes \left(H^*(\Omega S^2)\right)^k$$

extending the May-Segal formula [Seg73], [Ver99, Th. 8.11]

$$H^*(C_\infty(\mathbb{C}) \simeq H^*(\Omega^2 S^3).$$

Combining the results of Arnold and the cohomology of the loop spaces of a sphere

$$H^i(\Omega S^2) = \mathbb{Z}$$

for $i \geq 0$ [Hat04, Example 1.5]), this gives back corollary (2.2).

3. Poincaré Polynomials of $F_n(\mathbb{C} \setminus k)$

Arnold's calculation of $H^*(F_n(\mathbb{C}), \mathbb{Z})$ can be extended to $H^*(F_n(\mathbb{C} \setminus k), \mathbb{Z})$ via the fiber bundles

$$F_n(\mathbb{C} \setminus k) \mapsto F_{n-1}(\mathbb{C} \setminus k)$$

with fiber $\mathbb{C} \setminus (k + n - 1)$.

Theorem 3.1. *[Ver98, Thm. 7.1] We have*

$$P(F(\mathbb{C} \setminus k, n)) = (1 + kx)(1 + (k+1)x) \cdots (1 + (n+k-1)x).$$

4. Virtual Poincaré Polynomials of $F_n(\mathbb{C} \setminus k)$

We have

$$S(\mathbb{C} \setminus k) = S(\mathbb{CP}^1 \setminus k+1) = x^2 + 1 - (k+1) = x^2 - k.$$

Using the same fiber bundles or [Get95, Theorem, page 2] we get

Theorem 4.1. *The virtual Poincaré polynomials of $F_n(\mathbb{C} \setminus l)$ is given by*

$$S(F_n(\mathbb{C} \setminus k)) = (x^2 - k)(x^2 - k - 1) \cdots (x^2 - k - n + 1).$$

5. Virtual Poincaré Polynomials of $C_n(\mathbb{C} \setminus k)$

As $S(\mathbb{C} \setminus k) = (x^2 - k)$, the calculations of Getzler [Get95, Cor. 5.7] allow us to conclude

$$\sum_{n \geq 0} S(C_n(\mathbb{C} \setminus k))\, y^n = \frac{(1 - y^2 x^2)(1 - y)^k}{(1 - yx^2)(1 - y^2)^k},$$

which simplifies to

Theorem 5.1. *[Get95] The virtual Poincaré polynomials of $C_n(\mathbb{C} \setminus k)$ are given by the following generating series:*

$$\sum_{n \geq 0} S(C_n(\mathbb{C} \setminus k))\, y^n = \frac{(1 - y^2 x^2)}{(1 - yx^2)(1 + y)^k}$$

6. Comparision

We observe that under the variable transformation

$$x \to -1/x^2, y \to yx^2$$

the respective generating series

$$\sum_{n \geq 0} P(C_n(\mathbb{C} \setminus k))\, y^n \qquad\qquad \sum_{n \geq 0} P(F_n(\mathbb{C} \setminus k))\, y^n$$

transform into

$$\sum_{n \geq 0} S(C_n(\mathbb{C} \setminus k))\, y^n \qquad\qquad \sum_{n \geq 0} P(F_n(\mathbb{C} \setminus k))\, y^n.$$

This means, in this case the classical and virtual Poincaré polynomials are in some sense dual to each other.

Example 6.1. We look 3-pointed configuration spaces of $\mathbb{C} \setminus 2$:

$$P(C_3(\mathbb{C} \setminus 2)) = 4x^3 + 5x^2 + 3x + 1 \qquad P(F_3(\mathbb{C} \setminus 2)) = 24x^3 + 26x^2 + 9x + 1$$

$$S(C_3(\mathbb{C} \setminus 2) = x^6 - 3x^4 + 5x^2 - 4 \qquad S(F_3(\mathbb{C} \setminus 2)) = x^4 - 9x^4 + 26x^2 - 24$$

CHAPTER 6

Further Directions

We have seen that the explicit Betti numbers of configuration spaces can get quite complicated. So computing closed formulas for further cases might be possible, however it is not clear what one might learn from that. An example are the formulas of [DK16] for unordered configuration spaces of surfaces or the computations of Maguire [MCF16] for $C_n(\mathbb{C}P^n)$ for small m. However, the patterns of these formulas in n or g remain quite unclear. More structural insights seem to be necessary.

One of the most interesting applications of explicit calculations might be arithmetic questions. By the interpretation

$$C_n(\mathbb{C}) \simeq \{\text{ complex, monic, squarefree polynomials}\}$$

and the Grothendieck-Lefschetz fixed point theorem there is a relation between

$$\text{(weighted) counts of squarefree } \mathbb{Z}/p\mathbb{Z}\text{-polynomials of degree } n$$
$$=$$
$$S_n \text{ - representation theory of } H^*(F_n(\mathbb{C}), \mathbb{Q})$$

Analogous arithmetic interpretations exist for many other families of spaces with S_n-actions [CEF14]. For more complicated configuration spaces than $F_n(\mathbb{C})$ however, there seem to be no good tools to compute the multiplicities of other representations than the trivial one and only few examples have been done.

Bibliography

[AB14] S. Ashraf and B. Berceanu. “Cohomology of 3-points configuration spaces of complex projective spaces”. In: *Adv. Geom.* 14.4 (2014), pp. 691–718.

[Arn69] V. I. Arnold. “The cohomology ring of the colored braid group”. In: *Vladimir I. Arnold-Collected Works*. Springer, 1969, pp. 183–186.

[Arn70] V. I. Arnold. “On some topological invariants of algebraic functions”. In: *Vladimir I. Arnold-Collected Works*. Springer, 1970, pp. 199–221.

[Aza15] H. Azam. “Cohomology groups of configuration spaces of Riemann surfaces”. In: *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* 58(106).1 (2015), pp. 33–47.

[BC74] J. S. Birman and J. Cannon. *Braids, Links, and Mapping Class Groups. (AM-82)*. Princeton University Press, 1974.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. “The Magma algebra system. I. The user language”. In: *J. Symbolic Comput.* 24.3-4 (1997). Computational algebra and number theory (London, 1993), pp. 235–265.

[BG75] E. A. Bender and J. R. Goldman. “On the applications of Moebius inversion in combinatorial analysis”. In: *Amer. Math. Monthly* 82.8 (1975), pp. 789–803.

[BMP05] B. Berceanu, M. Markl, and S. Papadima. “Multiplicative models for configuration spaces of algebraic varieties”. In: *Topology* 44.2 (2005), pp. 415–440.

[CEF14] T. Church, J. Ellenberg, and B. Farb. “Representation stability in cohomology and asymptotics for families of varieties over finite fields”. In: *Contemporary Mathematics* 620 (2014), pp. 1–54.

[CEF15] T. Church, J. S. Ellenberg, and B. Farb. “FI-modules and stability for representations of symmetric groups”. In: *Duke Math. J.* 164.9 (2015), pp. 1833–1910.

[CF13] T. Church and B. Farb. “Representation theory and homological stability”. In: *Adv. Math.* 245 (2013), pp. 250–314.

[CG10] N. Chriss and V. Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Reprint of the 1997 edition. Birkhäuser Boston, Inc., Boston, MA, 2010, pp. x+495.

[Chu+14] T. Church et al. “FI-modules over Noetherian rings”. In: *Geom. Topol.* 18.5 (2014), pp. 2951–2984.

[Chu12] T. Church. “Homological stability for configuration spaces of manifolds”. In: *Invent. Math.* 188.2 (2012), pp. 465–504.

[CLM07] F. R. Cohen, T. J. Lada, and P. J. May. *The homology of iterated loop spaces*. Vol. 533. Springer, 2007.

[CP15] F. Cantero and M. Palmer. "On homological stability for configuration spaces on closed background manifolds". In: *Doc. Math.* 20 (2015), pp. 753–805.

[CT93] F. R. Cohen and L. R. Taylor. "On the representation theory associated to the cohomology of configuration spaces". In: *Algebraic topology (Oaxtepec, 1991)*. Vol. 146. Contemp. Math. Amer. Math. Soc., Providence, RI, 1993, pp. 91–109.

[DK16] G. Drummond-Cole and B. Knudsen. "Betti numbers of configuration spaces of surfaces". In: *ArXiv e-prints* (Aug. 2016). arXiv: `1608.07490`.

[FH01] E. R. Fadell and S. Y. Husseini. *Geometry and topology of configuration spaces*. Springer, 2001.

[FHT01] Y. Felix, S. Halperin, and J. Thomas. *Rational homotopy theory*. Vol. 205. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001.

[Fin47] N. J. Fine. "Binomial coefficients modulo a prime". In: *Amer. Math. Monthly* 54 (1947), pp. 589–592.

[FM94] W. Fulton and R. MacPherson. "A compactification of configuration spaces". In: *Annals of Mathematics* 139.1 (1994), pp. 183–225.

[FT00] Y. Felix and J.-C. Thomas. "Rational Betti numbers of configuration spaces". In: *Topology Appl.* 102.2 (2000), pp. 139–149.

[FT05] Y. Felix and D. Tanre. "The cohomology algebra of unordered configuration spaces". In: *J. London Math. Soc. (2)* 72.2 (2005), pp. 525–544.

[Fuk70] D. B. Fuks. "Cohomology of the braid group mod 2". In: *Funkcional. Anal. i Prilov zen.* 4.2 (1970), pp. 62–73.

[Get95] E. Getzler. "Mixed Hodge structures of configuration spaces". In: *eprint arXiv:alg-geom/9510018*. Oct. 1995.

[Get99] E. Getzler. "Resolving mixed Hodge modules on configuration spaces". In: *Duke Math. J.* 96.1 (1999), pp. 175–203.

[Hat02] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002.

[Hat04] A. Hatcher. *Spectral Sequences in Algebraic Topology*. 2004.

[KM16] A. Kupers and J. Miller. "Sharper periodicity and stabilization maps for configuration spaces of closed manifolds". In: *Proc. Amer. Math. Soc.* 144.12 (2016), pp. 5457–5468.

[Knu14] B. Knudsen. "Betti numbers and stability for configuration spaces via factorization homology". In: *ArXiv e-prints* (May 2014). arXiv: `1405.6696`.

[Kri94] I. Kriz. "On the rational homotopy type of configuration spaces". In: *Annals of Mathematics* 139.2 (1994), pp. 227–237.

[LS86] G. I. Lehrer and L. Solomon. "On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes". In: *J. Algebra* 104.2 (1986), pp. 410–424.

[McD75] D. McDuff. "Configuration spaces of positive and negative particles". In: *Topology* 14 (1975), pp. 91–107.

[MCF16] M. Maguire, w. A. b. M. Christie, and D. Francour. "Computing cohomology of configuration spaces". In: *ArXiv e-prints* (Dec. 2016). arXiv: `1612.06314`.

[Nag15] R. Nagpal. *FI-modules and the cohomology of modular representations of symmetric groups.* Thesis (Ph.D.)–The University of Wisconsin - Madison. ProQuest LLC, Ann Arbor, MI, 2015, p. 91.

[Nap03] F. Napolitano. “On the cohomology of configuration spaces on surfaces”. In: *J. Lond. Math. Soc.* 68.2 (02 2003), pp. 477–492.

[OS80] P. Orlik and L. Solomon. “Combinatorics and topology of complements of hyperplanes”. In: *Invent. Math.* 56.2 (1980), pp. 167–189.

[Ran13] O. Randal-Williams. “Topological chiral homology abd configuration spaces of spheres”. In: *Morfismos* 17.2 (2013), pp. 57–69.

[Sage] ”The Sage Developers”. *Sage Mathematics Software (Version 6.7)*. `sagemath.org`. 2015.

[Sal04] P. Salvatore. “Configuration spaces on the sphere and higher loop spaces”. In: *Math. Z.* 248.3 (2004), pp. 527–540.

[Sch16] C. Schiessl. “Betti numbers of unordered configuration spaces of the torus”. In: *ArXiv e-prints* (Feb. 2016). arXiv: `1602.04748`.

[Seg73] G. Segal. “Configuration-spaces and iterated loop-spaces”. In: *Invent. Math.* 21 (1973), pp. 213–221.

[Seg79] G. Segal. “The topology of spaces of rational functions”. In: *Acta Math.* 143.1-2 (1979), pp. 39–72.

[Sev84] M. B. Sevryuk. “The cohomology of projectively compactified complex swallowtails and their complements”. In: *Russian Mathematical Surveys* 39.5 (1984), p. 285.

[Sta11] R. P. Stanley. *Enumerative Combinatorics: Volume 1.* 2nd. New York, NY, USA: Cambridge University Press, 2011.

[Tot02] B. Totaro. “Topology of singular algebraic varieties”. In: *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002).* Higher Ed. Press, Beijing, 2002, pp. 533–541.

[Tot96] B. Totaro. “Configuration spaces of algebraic varieties”. In: *Topology* 35.4 (1996), pp. 1057–1067.

[Vai78] F. V. Vainshtein. “The cohomology of braid groups”. In: *Funktsional. Anal. i Prilozhen.* 12.2 (1978), pp. 72–73.

[Vas01] V. A. Vassiliev. *Introduction to topology.* Vol. 14. Student Mathematical Library. Translated from the 1997 Russian original by A. Sossinski. American Mathematical Society, Providence, RI, 2001, pp. xiv+149.

[Ver98] V. V. Vershinin. “Homology of braid groups and their generalizations”. In: *Knot theory (Warsaw, 1995).* Vol. 42. Banach Center Publ. Polish Acad. Sci. Inst. Math., Warsaw, 1998, pp. 421–446.

[Ver99] V. V. Vershinin. “Braid groups and loop spaces”. In: *Uspekhi Mat. Nauk* 54.2(326) (1999), pp. 3–84.